Classification of linearly compact simple rigid superalgebras

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Abstract

The notion of an anti-commutative (resp. commutative) rigid superalgebra is a natural generalisation of the notion of a Lie (resp. Jordan) superalgebra. Intuitively rigidity means that small deformations of the product under the structural group produce an isomorphic algebra. In this paper we classify all linearly compact simple anti-commutative (resp. commutative) rigid superalgebras. Beyond Lie (resp. Jordan) superalgebras the complete list includes four series and twenty two exceptional superalgebras (resp. ten exceptional superalgebras).

Introduction

The Killing-Cartan classification of simple finite-dimensional Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic 0 and Cartan's classification of simple infinite-dimensional Lie algebras of vector fields can be viewed as two parts of the classification of simple linearly compact Lie algebras. (Recall that a topological algebra is called linearly compact if the underlying topological vector space is linearly compact, i.e., it is isomorphic to the topological direct product of finite-dimensional vector spaces with the discrete topology). Thus, a complete list of simple linearly compact Lie algebras consists of classical finite-dimensional series: sl_m, so_m, sp_m (m even), five exceptional (finite-dimensional) Lie algebras: E_6, E_7, E_8, F_4, G_2, and four infinite-dimensional series of Lie algebras of formal vector fields: W_m, S_m, H_m (m even) and K_m (m odd).

This classification was extended to the Lie superalgebra case in [10] and [12]. A complete list of simple linearly compact Lie superalgebras consists of classical finite-dimensional series $sl(m,n)/\delta_{m,n}I$, $osp(m,n)$ (which include the classical Lie algebra series when m or n are 0), two "strange" series $p(n)$, $q(n)$, three exceptional finite-dimensional Lie superalgebras $D(2,1;\alpha)$, $F(4)$, $G(3)$, in addition to the five exceptional Lie algebras, eleven series of Lie superalgebras of vector fields (which are finite-dimensional if $m=0$): $W(m,n)$, $S(m,n)$, $H(m,n)$ (m even), $K(m,n)$ (m odd), $HO(m,m)$ ($m \geq 2$), $SHO(m,m)$ ($m \geq 3$), $KO(m,m+1)$ ($m \geq 1$), $SKO(m,m+1;\beta)$ ($m \geq 2$), $\tilde{S}(0,n)$ ($n \geq 4$, even), $SHO^\sim(m,m)$ ($m \geq 2$, even), $SKO^\sim(m,m+1)$ ($m \geq 3$, odd) (which include the Lie algebra series $W_m = W(m,0)$, $S_m = S(m,0)$, $H_m = H(m,0)$, $K_m = K(m,0)$), and five exceptional infinite-dimensional Lie superalgebras: $E(1,6)$, $E(3,6)$, $E(3,8)$, $E(4,4)$, $E(5,10)$.

Closely related, via the Tits-Kantor-Koecher (TKK) construction, is the classification of simple linearly compact Jordan superalgebras, obtained in [4]. It is based on the observation that isomorphism classes of such Jordan superalgebras correspond bijectively to conjugacy classes of short sl_2 subalgebras in the Lie superalgebra $DerL$ of continuous derivations of a simple linearly compact

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Lie superalgebra L. Recall that an sl_2 subalgebra of a Lie superalgebra is called short if it has a basis e, h, f, such that $[h,e] = -e$, $[h,f] = f$, $[e,f] = h$ and the only eigenvalues of $ad\,h$ on L are 0,1 or -1. Given such a subalgebra in $DerL$, the corresponding simple Jordan superalgebra is the -1-eigenspace J of $ad\,h$ in L, endowed with the product

$$a \circ b = [[f,a],b], \quad a,b \in J. \tag{0.1}$$

The complete list of simple linearly compact Jordan superalgebras consists of classical finite-dimensional series $gl(m,n)_+$, $osp(m,n)_+$ (n even), $(m,n)_+$ (n even), two "strange" series $p(n)_+$, $q(n)_+$, the series $JP(m,n)$ (which is finite-dimensional if $m=0$), four finite-dimensional exceptional superalgebras E, F, D_t, K, and two exceptional infinite-dimensional superalgebras, JCK and JS. The finite-dimensional part of this classification goes back to [1] and [11] in the non-super and super cases respectively.

The constructions of finite-dimensional classical and "strange" series of Lie and Jordan superalgebras are parallel and very simple. Recall that all finite-dimensional associative simple superalgebras consist of two series: $mat(m,n)$ (the superalgebra of endomorphisms of the superspace $\mathbb{F}^{m|n}$) and, in the case $m=n$, its subalgebra $qmat(n,n)$, consisting of matrices commuting with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Given an associative superalgebra A, one denotes by A_- (resp. A_+) the space A with the product

$$[a,b] = ab - (-1)^{p(a)p(b)}ba \qquad (\text{resp. } a \circ b = ab + (-1)^{p(a)p(b)}ba). \tag{0.2}$$

Here and further, $p(a) \in \mathbb{Z}/2\mathbb{Z}$ denotes the parity of $a \in A$. Then $sl(m,n)/\delta_{m,n}I$ (resp. $q(n)$) is the derived algebra divided by the center for $mat(m,n)_-$ (resp. $qmat(n,n)_-$) and $gl(m,n)_+ = mat(m,n)_+$. The Lie (resp. Jordan) superalgebra $osp(m,n)$ (resp. $osp(m,n)_+$) is the subalgebra of skewsymmetric (resp. symmetric) endomorphisms in $mat(m,n)_-$ (resp. $mat(m,n)_+$) with respect to a non-degenerate even supersymmetric bilinear form on $\mathbb{F}^{m|n}$. The Lie (resp. Jordan) superalgebra $p(n)$ (resp. $p(n)_+$) is the derived algebra divided by the center for the subalgebra of skew-symmetric endomorphisms in $mat(n,n)$ (resp. the subalgebra of symmetric endomorphisms) with respect to an odd non-degenerate bilinear form on $\mathbb{F}^{n|n}$. Finally, $(m,n)_+ = \mathbb{F}e \oplus \mathbb{F}^{m|n}$, where e is the identity element and $a \circ b = (a|b)e$ for $a,b \in \mathbb{F}^{m|n}$, where $(\cdot|\cdot)$ is an even non-degenerate supersymmetric bilinear form on $\mathbb{F}^{m|n}$.

The series $JP(m,n)$ is obtained from the generalized Poisson superalgebras $P(m,n)$ via the following Kantor-King-McCrimmon (KKM) construction [13], [14]. Recall [4] that a generalized Poisson superalgebra P is a unital commutative associative superalgebra, endowed with a Lie superalgebra bracket $\{\cdot,\cdot\}$, satisfying the generalized Leibniz rule

$$\{a,bc\} = \{a,b\}c + (-1)^{p(a)p(b)}b\{a,c\} + D(a)bc, \tag{0.3}$$

where $D(a) = \{1,a\}$. The associated Jordan superalgebra is $JP = P \oplus \bar{P}$, where $\bar{P}$ is a copy of P with reversed parity, endowed with the following Jordan product, where $a,b \in P$, $\bar{a},\bar{b} \in \bar{P}$:

$$a \circ b = ab, \quad \bar{a} \circ b = \overline{ab}, \quad a \circ \bar{b} = (-1)^{p(a)}\overline{ab}, \quad \bar{a} \circ \bar{b} = (-1)^{p(a)}\{a,b\}_D, \tag{0.4}$$

where $\{a,b\}_D = \{a,b\} - \frac{1}{2}(aD(b) - D(a)b)$.

The KKM construction allows one to derive a classification of simple linearly compact generalized Poisson superalgebras from that of simple linearly compact Jordan superalgebras [4]. The

result is that, up to gauge equivalence, any such generalized Poisson superalgebra is one of the $P(m,n)$. Recall that a bracket $\{a,b\}^{\varphi}=\varphi^{-1}\{\varphi a,\varphi b\}$, where φ is an even invertible element, is called gauge equivalent to the bracket $\{\cdot,\cdot\}$. Also, the generalized Poisson superalgebra $P(m,n)$ is the associative commutative superalgebra of formal power series in the even indeterminates $p_1,\dots,p_k,q_1,\dots,q_k$ (resp. $p_1,\dots,p_k,q_1,\dots,q_k,t$) if $m=2k$ (resp. $m=2k+1$) and in the n odd indeterminates $\xi_1,\dots,\xi_n$, endowed with the bracket:

$$\{a,b\}=\sum_{i=1}^{k}(\frac{\partial a}{\partial p_i}\frac{\partial b}{\partial q_i}-\frac{\partial a}{\partial q_i}\frac{\partial b}{\partial p_i})+(-1)^{p(a)}\sum_{i=1}^{n}\frac{\partial a}{\partial \xi_i}\frac{\partial b}{\partial \xi_i}+(2-E)(a)\frac{\partial b}{\partial t}-\frac{\partial a}{\partial t}(2-E)(b), \tag{0.5}$$

where $E=\sum_{i=1}^{k}(p_i\frac{\partial}{\partial p_i}+q_i\frac{\partial}{\partial q_i})+\sum_{i=1}^{n}\xi_i\frac{\partial}{\partial \xi_i}$ (the last two terms in (0.5) vanish if m is even).

Our first basic idea is to generalize the notion of a Jordan superalgebra as follows. Note that a commutative superalgebra is a vector superspace J together with an even element μ of $Hom(S^2J,J)$, so that $a\circ b=\mu(a\otimes b)$. Define the structural Lie superalgebra $Str(J,\mu)$ as a subalgebra of the Lie superalgebra $End(J,J)_-$ generated by all operators of left multiplication $\mu_a(b)=\mu(a\otimes b)$, $a\in J$, and denote by $R(J,\mu)$ the minimal submodule of the $Str(J,\mu)$-module $Hom(S^2J,J)$, containing μ. One of the basic properties of a Jordan superalgebra is [11]

$$R(J,\mu)=Str(J,\mu)\mu+\mathbb{F}\mu. \tag{0.6}$$

Note that in the case J is a finite-dimensional Jordan algebra, $R(J,\mu)$ is the span of the orbit of $\mathbb{F}\mu$ under the structural group, and property (0.6) means that this orbit is dense in $R(J,\mu)$. Thus, (0.6) means a certain rigidity property, namely, "small" deformations of the product by the structural group produce an isomorpic algebra.

We define a *commutative rigid superalgebra* to be a commutative superalgebra (J,μ), satisfying (0.6). The first result of our paper is

Theorem 0.1 *Any simple linearly compact commutative rigid superalgebra, which is not a Jordan superalgebra, is isomorphic either to one of the five infinite-dimensional superalgebras $JS_{1,1}$, $JSHO_{2,2}$, $JSKO_{1,2}$, $JS_{1,8}^{\alpha}$ for $\alpha=0,1$, or to one of the five finite-dimensional superalgebras $JS_{0,2}$, $JW_{0,4}$, $JW_{0,8}$, $JS_{0,8}$, $JS_{0,16}$.*

In order to explain the construction of the superalgebras appearing in the statement of Theorem 0.1, introduce some notation. Given a commutative associative superalgebra A and its derivation D, denote by AD the vector superspace of derivations of A of the form fD, where $f\in A$. Given even derivations D_i of A and elements $f_i\in A$ $(i=1,2)$, let

$$[f_1D_1\,,\,f_2D_2]_{\pm}=f_1D_1(f_2)D_2\pm(-1)^{p(f_1)p(f_2)}f_2D_2(f_1)D_1. \tag{0.7}$$

Given odd derivations D_i of A and elements $f_i\in A$ $(i=1,2)$, let

$$[f_1D_1\,,\,f_2D_2]_{+}=f_1D_1(f_2)D_2+(-1)^{p(f_1)+1)(p(f_2)+1)}f_2D_2(f_1)D_1 \tag{0.8}$$

Recall that the product $[fD,gD]_+$ defines on AD a Jordan superalgebra structure for any odd derivation D of A [4, Example 4.6].

Our first example is $JS_{1,1}=\mathbb{F}[[x]]\frac{d}{dx}$ with the product

$$f\frac{d}{dx}\circ g\frac{d}{dx}=\left[f\frac{d}{dx}\,,\,g\frac{d}{dx}\right]_+, \tag{0.9}$$

which we call the Beltrami algebra.

We shall denote by $\bar{A}$ a copy of A with reversed parity, and by $\bar{a} \in \bar{A}$ the element corresponding to $a \in A$.

The next example is $JSHO_{2,2} = \mathbb{F}[[p,q]]\frac{\partial}{\partial p} \oplus \overline{\mathbb{F}[[p,q]]}$ with the product $(f, g \in \mathbb{F}[[p,q]]\,,\ \bar{f}, \bar{g} \in \overline{\mathbb{F}[[p,q]]})$:

$$f\frac{\partial}{\partial p} \circ g\frac{\partial}{\partial p} = \Big[f\frac{\partial}{\partial p}, g\frac{\partial}{\partial p}\Big]_+ \,,\ f\frac{\partial}{\partial p} \circ \bar{g} = \overline{f\frac{\partial g}{\partial p}}\,,\ \bar{f} \circ \bar{g} = \Big(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\Big)\frac{\partial}{\partial p}\,. \tag{0.10}$$

The next example is $JSKO_{1,2} = \mathbb{F}[[x]]\frac{d}{dx} \oplus \overline{\mathbb{F}[[x]]}$ with the product $(f, g \in \mathbb{F}[[x]]\,,\ \bar{f}, \bar{g} \in \overline{\mathbb{F}[[x]]})$

$$f\frac{d}{dx} \circ g\frac{d}{dx} = \Big[f\frac{d}{dx}\,,\ g\frac{d}{dx}\Big]_+ \,,\ f\frac{d}{dx} \circ \bar{g} = \overline{f\frac{dg}{dx}}\,,\ \bar{f} \circ \bar{g} = 2\Big[f\frac{d}{dx}\,,\ g\frac{d}{dx}\Big]_-\,. \tag{0.11}$$

The remaining two infinite-dimensional examples are $JS^{\alpha}_{1,8}$ for $\alpha = 0, 1$, constructed as follows. Let $A = \mathbb{F}[[x, \xi_1, \xi_2]]$, where ξ_1 and ξ_2 are Grassmann indeterminates, and let

$$D_1 = \frac{\partial}{\partial \xi_1} + \xi_1\frac{\partial}{\partial x} + \alpha\xi_2\frac{\partial}{\partial x}\,,\ D_2 = \frac{\partial}{\partial \xi_2} + x\xi_2\frac{\partial}{\partial x} - \xi_1\xi_2\frac{\partial}{\partial \xi_1}\,.$$

Then $JS^{\alpha}_{1,8} = AD_1 \oplus AD_2$, with the product $(f, g \in A)$:

$$fD_i \circ gD_j = [fD_i, gD_j]_+\,, \quad i, j = 1, 2\,. \tag{0.12}$$

The four finite-dimensional superalgebras $JW_{0,4}$, $JW_{0,8}$, $JS_{0,8}$ and $JS_{0,16}$ are constructed in a similar way (see Example 4.3). Finally, $JS_{0,2} = \mathbb{F}a + \mathbb{F}b$ with products: $a^2 = b$, $b^2 = a$, $a \circ b = 0$.

Our second basic idea is to study odd type superalgebras, aimed at the classification of odd generalized Poisson superalgebras. Odd Poisson superalgebras have appeared both in mathematics and physics literature, like the work of Gerstenhaber on Hochshild cohomology, and the work on the Batalin-Vilkovisky quantization.

By definition, an *odd type superalgebra* is a superalgebra with reversed parity. Equivalently, it is (J, μ) with an odd $\mu \in Hom(S^2J, J)$. The advantage of this notion is that, given an anti-commutative superalgebra A with product $\bullet$, reversing the parity of A and defining the new product

$$a \circ b = (-1)^{p(a)} a \bullet b, \tag{0.13}$$

produces a commutative odd type superalgebra $\bar{A}$, and viceversa, reversing the parity, we pass from a commutative odd type superalgebra to an anti-commutative superalgebra.

An *odd generalized Poisson superalgebra* is defined as a unital associative commutative superalgebra, endowed with a bracket $\{\cdot,\cdot\}$, which is a Lie superalgebra bracket with respect to reversed parity, such that the following generalized odd Leibniz rule holds:

$$\{a, bc\} = \{a, b\}c + (-1)^{(p(a)+1)p(b)} b\{a, c\} + (-1)^{p(a)+1} D(a)bc, \tag{0.14}$$

where $D(a) = \{1, a\}$. An *odd Poisson superalgebra* is a special case of this, when $D = 0$.

The basic examples of odd generalized Poisson superalgebras are $PO(n, n)$ (resp. $PO(n, n+1)$), which are associative commutative superalgebras of formal power series in n even indeterminates

$x_1, \dots, x_n$ and n (resp. $n+1$) odd indeterminates $\xi_1, \dots, \xi_n$ (resp. $\xi_1, \dots, \xi_n, \tau$), endowed with the following bracket:

$$(0.15) \qquad \{a,b\} = \sum_{i=1}^{n} \Big(\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial \xi_i} + (-1)^{p(a)}\frac{\partial a}{\partial \xi_i}\frac{\partial b}{\partial x_i}\Big) + (E-2)(a)\frac{\partial b}{\partial \tau} + (-1)^{p(a)}\frac{\partial a}{\partial \tau}(E-2)(b),$$

where $E = \sum_{i=1}^{n}(x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i})$ (in $PO(n,n)$ case the last two terms in (0.15) vanish). Note also that $PO(n,n)$ is an odd Poisson superalgebra.

We define a *commutative rigid odd type superalgebra* to be an odd type commutative superalgebra (J,μ), where μ is an odd element of $Hom(S^2J, J)$, satisfying condition (0.6). This is equivalent, by reversing the parity and redefining the product as in (0.13), to the notion of an anti-commutative rigid superalgebra.

Note that any Lie superalgebra $\mathfrak{g}$ is an anti-commutative rigid superalgebra. Indeed, it is easy to see that $Str(\mathfrak{g},\mu) = \mathfrak{g}$ and $R(\mathfrak{g},\mu) = \mathbb{F}\mu$. Hence any Lie superalgebra is an anti-commutative rigid superalgebra. It turns out that there are many more anti-commutative rigid superalgebras. At least one of these algebras appears in the theory of PDEs under the name Jacobi-Mayer bracket, which is given on the factorspace by $\mathbb{F}1$ of the space of formal power series in x, y, z by the following formula (see [7]):

$$\{f,g\} = \det \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ 0 & -x & 1 \end{pmatrix}.$$

Our second main result is

Theorem 0.2 *Any simple linearly compact anti-commutative rigid superalgebra, which is not a Lie superalgebra, is isomorphic either to one of the four series, described by Examples 5.1, 5.4, and 5.5, related to odd generalized Poisson superalgebras, or to one of the twenty two exceptions, described by Examples 5.6–5.11, 5.13–5.15, 5.17–5.18, 5.20, 5.22–5.24.*

The proof of Theorem 0.1 and Theorem 0.2 goes as follows. Given a linearly compact vector superspace J, consider the universal $\mathbb{Z}$-graded Lie superalgebra $W(J) = \prod_{k=-1}^{\infty} W_k(J)$. *Universality* means that any transitive $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \prod_{k=-1}^{\infty} \mathfrak{g}_k$ with $\mathfrak{g}_{-1} = J$ canonically embeds in $W(J)$, and *transitivity* means that $[x, \mathfrak{g}_{-1}] = 0$, $x \in \mathfrak{g}_k$, $k \geq 0$, implies $x = 0$. Then $W_k(J) = Hom(S^{k+1}(J), J)$, so that any even (resp. odd) element $\mu \in W_1(J)$ defines a commutative superalgebra (resp. commutative odd type superalgebra) structure on J and viceversa. Hence to any commutative superalgebra (resp. odd type superalgebra) structure on J we can canonically associate the Lie superalgebra $Lie(J,\mu) = \prod_{k=-1}^{\infty} \mathfrak{g}_k$, which is the graded subalgebra of $W(J)$, generated by J and μ. We thus obtain a transitive $\mathbb{Z}$-graded Lie superalgebra $Lie(J,\mu)$ such that the following two properties hold:

$$(0.16) \qquad \mathfrak{g}_k = \mathfrak{g}_1^k \ \ for\ k \geq 1,$$

$$(0.17) \qquad [\mathfrak{g}_{-1}, \mu] \ \text{ generates the Lie superalgebra } \mathfrak{g}_0.$$

As a result, the classification of algebra structures on J is equivalent to the classification of $\mathbb{Z}$-graded transitive Lie superalgebras $\mathfrak{g} = \prod_{k=-1}^{\infty} \mathfrak{g}_k$ together with an element $\mu \in \mathfrak{g}_1$, satisfying (0.16) and

(0.17). We call this the *generalized TKK construction.* The simplicity of the superalgebra J is equivalent to the following *irreducibility* of the grading:

$$\text{the } \mathfrak{g}_0-\text{module } \mathfrak{g}_{-1} \text{ is irreducible.} \tag{0.18}$$

Thus, the classification of simple algebra structures on J is equivalent to the classification of all transitive irreducible $\mathbb{Z}$-graded Lie superalgebras (of depth 1) together with $\mu \in \mathfrak{g}_1$, satisfying (0.16) and (0.17). Commutative (resp anti-commutative) rigid superalgebras correspond to such $\mathbb{Z}$-graded superalgebras with $\mu \in \mathfrak{g}_1$, satisfying property (0.6), where μ is even (resp. odd). This reduces the classification of simple linearly compact commutative (resp. anti-commutative) rigid superalgebras to the classification of linearly compact transitive $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g} = \prod_{k=-1}^{\infty} \mathfrak{g}_k$, satisfying (0.16), (0.17), (0.18) and (0.6) for some even (resp. odd) element $\mu \in \mathfrak{g}_1$. Such a $\mathbb{Z}$-grading is called *admissible.*

We obtain a classification of the latter in two steps. First, showing that $\mathfrak{g}$ is close to being simple and using the classification of simple and semisimple linearly compact Lie superalgebras obtained in [12] and [3], respectively, we obtain a classification of all linearly compact Lie superalgebras which may have an admissible grading. Next, from the classification of all gradings, obtained in [3], we extract the classification of all admissible gradings.

However this still does not complete the classification of linearly compact simple commutative (resp. anti-commutative) rigid superalgebras since we still have to classify, for each admissible grading, all even (resp. odd) $\mu \in \mathfrak{g}_1$, satisfying (0.17) and (0.6). If $\dim \mathfrak{g}_1 < \infty$, property (0.6) implies that for the algebraic group G, whose Lie algebra is the even part of $\mathfrak{g}_0$, we have, if μ is even (resp. odd): $G \cdot \mu$ is an open orbit in the even (resp. odd) part of $\mathfrak{g}_1$. Hence, up to isomorphism, there exists a unique μ in question.

If $\dim \mathfrak{g}_1 = \infty$, the situation is more complicated, since only a subalgebra of finite codimension n of the even part of $\mathfrak{g}_0$ can be integrated to a group, which we again denote by G. Hence we have to classify the orbits of G in the even (resp. odd) part of $\mathfrak{g}_1$ of codimension at most n (satisfying the additional property (0.17)).

The somewhat awkward notation for a rigid superalgebra (J, μ) is derived from that of the associated Lie superalgebra $\mathfrak{g} = Lie(J, \mu)$ by adding J (resp. L) in the commutative (resp. anti-commutative) case in front of the notation for the type of $\mathfrak{g}$, the first and the second index in the notation being the growth and the size of J (see Section 6 for their definition).

A posteriori it turns out that the even and odd parts of a simple commutative or anti-commutative rigid linearly compact superalgebra have equal growth (resp. size), unless its odd part is zero.

In conclusion, we use the classification of odd type simple rigid commutative linearly compact superalgebras to derive the classification of simple linearly compact odd generalized Poisson superalgebras: all of them turn out to be gauge equivalent to $PO(n, n)$ or $PO(n, n + 1)$.

Throughout the paper the base field $\mathbb{F}$ is algebraically closed of characteristic 0.

1 Preliminaries on superalgebras

Definition 1.1 *A superalgebra (resp. odd type superalgebra) is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$, endowed with a product $a \otimes b \mapsto ab$, such that $ab \in A_{p(a)+p(b)}$ (resp. $A_{p(a)+p(b)+\bar{1}}$). In this case the product is called even (resp. odd).*

Definition 1.2 *A superalgebra or an odd type superalgebra, is called commutative (resp. anti-commutative) if $ab = (-1)^{p(a)p(b)}ba$ (resp. $ab = -(-1)^{p(a)p(b)}ba$).*

Remark 1.3 Reversing parity exchanges superalgebras with odd type superalgebras.

Remark 1.4 Let (R, μ) be a commutative (resp. anti-commutative) superalgebra with the product μ. Define the new product $\bar{\mu}(a \otimes b) = (-1)^{p(a)} \mu(a \otimes b)$ and let $\bar{R}$ be the vector superspace, obtained from R by reversing the parity. Then $(\bar{R}, \bar{\mu})$ is an odd type anti-commutative (resp. commutative) superalgebra, and viceversa.

A *linearly compact superalgebra* is a topological superalgebra whose underlying topological space is linearly compact. For a discussion on linearly compact vector spaces and algebras we refer to [4, §2], restricting ourselves here to a few remarks. Any finite-dimensional vector space is linearly compact. The basic example of a commutative associative linearly compact superalgebra is the superalgebra $\mathcal{O}(m,n) = \Lambda(n)[[x_1, \dots, x_m]]$ with the formal topology, where $\Lambda(n)$ denotes the Grassmann algebra over $\mathbb{F}$ on n anti-commuting indeterminates $\xi_1, \dots, \xi_n$, and the superalgebra parity is defined by $p(x_i) = \bar{0}$, $p(\xi_j) = \bar{1}$. The basic example of a linearly compact Lie superalgebra is $W(m,n) = Der\mathcal{O}(m,n)$, the Lie superalgebra of all continuous derivations of the superalgebra $\mathcal{O}(m,n)$. One has:

$$W(m,n) := \{X = \sum_{i=1}^{m} P_i(x,\xi) \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j(x,\xi) \frac{\partial}{\partial \xi_j} \mid P_i, Q_j \in \mathcal{O}(m,n)\}.$$

Letting $a_i = \deg x_i = -\deg \frac{\partial}{\partial x_i} \in \mathbb{Z}_{>0}$ and $b_j = \deg \xi_j = -\deg \frac{\partial}{\partial \xi_j} \in \mathbb{Z}$ we obtain a $\mathbb{Z}$-grading of finite depth of the Lie superalgebra $W(m,n)$, called the grading of type $(a_1, \dots, a_m | b_1, \dots, b_n)$ (cf. [12, Example 4.1]). When such a grading induces a $\mathbb{Z}$-grading on a closed subalgebra S of $W(m,n)$, the induced grading on S is also called a grading of type $(a_1, \dots, a_m | b_1, \dots, b_n)$. The grading of $W(m,n)$ of type $(1, \dots, 1 | 1, \dots, 1)$ is called the principal grading.

The following is a generalization of $W(m,n)$ to the case of an infinite number of indeterminates. Let J be a vector superspace and let $W_k(J)$ be the superspace of $(k+1)$-linear supersymmetric functions on J with values in J, i.e., $f \in W_k(J)$ if $f(\dots, x, y, \dots) = (-1)^{p(x)p(y)} f(\dots, y, x, \dots)$. Let $W(J) = \prod_{k=-1}^{\infty} W_k(J)$. We endow this vector superspace with a bracket as follows. If $f \in W_p(J)$, $g \in W_q(J)$, then we define $f \Box g$ to be the following element in $W_{p+q}(J)$:

$$f \Box g(x_0, \dots, x_{p+q}) = \sum_{\substack{i_0 < \dots < i_q \\ i_{q+1} < \dots < i_{p+q}}} \epsilon(i_0, \dots, i_q, i_{q+1}, \dots, i_{p+q}) f(g(x_{i_0}, \dots, x_{i_q}), x_{i_{q+1}}, \dots, x_{i_{p+q}}).$$

Here $\epsilon = (-1)^N$, where N is the number of interchanges of indeces of odd x_i's in the permutation.

Proposition 1.5 *The bracket*

$$[f, g] = f \Box g - (-1)^{p(f)p(g)} g \Box f \tag{1.1}$$

defines a Lie superalgebra structure on $W(J)$.

Proof. It is clear that $f \Box g$ is supersymmetric if both f and g are. The anti-commutativity of bracket (1.1) is immediate. Finally, it is easy to see that the operation $\Box$ is right associative, i.e., $(a, b, c) = (-1)^{p(b)p(c)}(a, c, b)$, where $(a, b, c) = (a \Box b) \Box c - a \Box (b \Box c)$. The right associativity of $\Box$ implies the Jacobi identity for bracket (1.1). $\Box$

Remark 1.6 If J is a vector superspace of dimension $(m|n)$, then $W(J)$ is isomorphic to the Lie superalgebra $W(m,n)$ [11, §2.1].

Remark 1.7 According to the above definitions, $W_0(J) = End(J)$, acting on $W_{-1}(J) = J$ in the obvious way and on $W_1(J)$ as follows: for $f \in W_0(J)$, $B \in W_1(J)$ and $x, y \in W_{-1}(J)$,

$$[f, B](x, y) = f(B(x, y)) - (-1)^{p(f)p(B)}(B(f(x), y) + (-1)^{p(x)p(y)}B(f(y), x)).$$

Besides, for $A \in W_k(J)$ and $x, y_i \in W_{-1}(J)$, $i = 1, ..., k$, we have: $[A, x](y_1, ..., y_k) = A(x, y_1, ..., y_k)$.

2 Odd generalized Poisson superalgebras

Definition 2.1 *An odd generalized Poisson superalgebra is a superalgebra P with two operations: a unital commutative associative superalgebra product $a \otimes b \mapsto ab$, and a Lie superalgebra bracket $a \otimes b \mapsto \{a, b\}$ with respect to reversed parity, satisfying the generalized odd Leibniz rule, namely, for $a, b, c \in P$ one has:*

$$\{a, bc\} = \{a, b\}c + (-1)^{(p(a)+1)p(b)}b\{a, c\} + (-1)^{p(a)+1}D(a)bc \tag{2.1}$$

where $D(a) = \{e, a\}$, e being the unit element. If $D = 0$, then relation (2.1) becomes the odd Leibniz rule; in this case P is called an odd Poisson superalgebra. (Note that D is an odd derivation of the associative product and of the Lie superalgebra bracket.)

Example 2.2 Consider the associative superalgebra $\mathcal{O}(n, n)$ with the following bracket, known as the Buttin bracket ($f, g \in \mathcal{O}(n, n)$):

$$\{f, g\}_H = \sum_{i=1}^{n}(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i}). \tag{2.2}$$

Then $\mathcal{O}(n, n)$ with this bracket is an odd Poisson superalgebra, which we denote by $PO(n, n)$.

Example 2.3 Consider the associative superalgebra $\mathcal{O}(n, n+1)$ with even indeterminates $x_1, \ldots, x_n$ and odd indeterminates $\xi_1, \ldots, \xi_n, \xi_{n+1} = \tau$. Define on $\mathcal{O}(n, n+1)$ the following bracket ($f, g \in \mathcal{O}(n, n+1)$):

$$\{f, g\}_K = \{f, g\}_H + (E - 2)(f)\frac{\partial g}{\partial \tau} + (-1)^{p(f)}\frac{\partial f}{\partial \tau}(E - 2)(g), \tag{2.3}$$

where $\{\cdot, \cdot\}_H$ is the Buttin bracket (2.2) and $E = \sum_{i=1}^{n}(x_i\frac{\partial}{\partial x_i} + \xi_i\frac{\partial}{\partial \xi_i})$ is the Euler operator. Then $\mathcal{O}(n, n+1)$ with bracket $\{\cdot, \cdot\}_K$ is an odd generalized Poisson superalgebra with $D = -2\frac{\partial}{\partial \tau}$ [3, Remark 4.1], which we denote by $PO(n, n+1)$.

Odd generalized Poisson superalgebras give rise to several series of simple linearly compact Lie superalgebras. Consider $\mathcal{O}(n, n)$ with its Lie superalgebra structure defined by bracket (2.2). Then $\mathbb{F}1$ is a central ideal of this Lie superalgebra, hence $HO(n, n) := \mathcal{O}(n, n)/\mathbb{F}1$, with reversed parity, is a linearly compact Lie superalgebra [5, §1.3]. The Lie superalgebra $HO(n, n)$ is simple if and only if $n \geq 2$ (cf. [12, Example 4.6]).

The Lie superalgebra $HO(n,n)$ contains the subalgebra $SHO'(n,n) = \{f \in \mathcal{O}(n,n)/\mathbb{F}1 \mid \Delta(f) = 0\}$ where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial \xi_i}$ is the odd Laplacian. For $n \geq 2$ the derived algebra of $SHO'(n,n)$ is an ideal of codimension 1, denoted by $SHO(n,n)$, consisting of elements not containing the monomial $\xi_1 \dots \xi_n$ [5, §1.3]. $SHO(n,n)$ is simple if and only if $n \geq 3$ [12, Example 4.7].

We shall denote by $KO(n,n+1)$ the superalgebra $PO(n,n+1)$ with its Lie superalgebra structure and reversed parity. This is a simple linearly compact Lie superalgebra for every $n \geq 1$ (cf. [12, Example 4.8]). Since $KO(1,2) \cong W(1,1)$ (cf. [12, Remark 6.6]), when dealing with the Lie superalgebra $KO(n,n+1)$ we shall always assume $n \geq 2$.

For $\beta \in \mathbb{F}$, let $div_\beta = \Delta + (E - n\beta)\frac{\partial}{\partial \tau}$. The Lie superalgebra $KO(n,n+1)$ contains the subalgebra $SKO'(n,n+1;\beta) = \{f \in \mathcal{O}(n,n+1) \mid div_\beta f = 0\}$. Let $SKO(n,n+1;\beta)$ be the derived algebra of $SKO'(n,n+1;\beta)$. Then $SKO(n,n+1;\beta)$ is simple for $n \geq 2$ and coincides with $SKO'(n,n+1;\beta)$ unless $\beta = 1$ or $\beta = \frac{n-2}{n}$. The Lie superalgebra $SKO(n,n+1;1)$ (resp. $SKO(n,n+1;\frac{n-2}{n})$) consists of the elements of $SKO'(n,n+1;1)$ (resp. $SKO'(n,n+1;\frac{n-2}{n})$) not containing the momomial $\tau\xi_1 \dots \xi_n$ (resp. $\xi_1 \dots \xi_n$) [5, §1.4].

Example 2.4 Let P be a unital odd generalized Poisson superalgebra with bracket $\{\cdot,\cdot\}$ and derivation D. Then for any even invertible $\varphi \in P$, formula

$$\{a,b\}^\varphi = \varphi^{-1}\{\varphi a, \varphi b\}$$

defines another odd generalized Poisson bracket on P if and only if $\{\varphi,\varphi\} = 0$. In this case the derivation corresponding to $\{\cdot,\cdot\}^\varphi$ is $D^\varphi(a) = -D(\varphi)a + \{\varphi, a\}$. We denote this odd generalized Poisson superalgebra by P^φ. The odd generalized Poisson superalgebras P and P^φ are called *gauge equivalent*.

3 Rigid superalgebras

Let J be a vector superspace. Then products (resp. odd type products) on J that make it a superalgebra (resp. odd type superalgebra) correspond bijectively to even (resp. odd) elements μ in the vector superspace $Hom(J \otimes J, J)$, so that $a \circ b = \mu(a \otimes b)$ is the product, corresponding to μ. Denote by $\mu_a \in End(J)$ the operator of left multiplication by $a \in J$ with respect to this product: $\mu_a(b) = a \circ b$.

Definition 3.1 *The structural Lie superalgebra $Str(J,\mu)$ for the superalgebra (J,μ) is the subalgebra of the Lie superalgebra $End(J)_-$, generated by all μ_a with $a \in J$.*

Note that the vector superspace $Hom(J \otimes J, J)$ is naturally a module over the Lie superalgebra $(End J)_-$, hence over its subalgebra $Str(J,\mu)$. The action is defined as follows ($f \in (End J)_-$, $B \in Hom(J \otimes J, J)$, $x, y \in J$):

$$[f,B](x,y) = f(B(x,y)) - (-1)^{p(f)p(B)}(B(f(x),y) + (-1)^{p(x)p(f)}B(x,f(y))). \tag{3.1}$$

Definition 3.2 *The space of related products $R(J,\mu)$ for the superalgebra (J,μ) is the minimal $Str(J,\mu)$-submodule of $Hom(J \otimes J, J)$, containing μ.*

In the case J is linearly compact we define $Str(J,\mu)$ and $R(J,\mu)$ as closures of the corresponding subspaces.

Definition 3.3 *A vector superspace J with a product (resp. odd type product) μ is called a rigid superalgebra if μ is an even (resp. odd) element of $Hom(J \otimes J, J)$, such that*

$$R(J, \mu) = Str(J, \mu)(\mu) + \mathbb{F}\mu.$$

Example 3.4 Any Jordan superalgebra is a commutative rigid superalgebra (see [11, Remark 6]).

Example 3.5 Any Lie superalgebra $\mathfrak{g}$ is an anti-commutative rigid superalgebra with $Str(\mathfrak{g}, \mu) = \mathfrak{g}$ and $R(\mathfrak{g}, \mu) = \mathbb{F}\mu$.

Other examples of rigid commutative and anti-commutative superalgebras will be given in Sections 4 and 5 respectively (see also the introduction).

Remark 3.6 If (J, μ) is a commutative superalgebra, then μ is an even element of $W_1(J) = Hom(S^2(J), J)$, and $R(J, \mu)$ is a $Str(J, \mu)$-submodule of $W_1(J)$. If (J, μ) is an anti-commutative superalgebra, then the odd type superalgebra $(\bar{J}, \bar{\mu})$, defined in Remark 1.4, is commutative, $\bar{\mu}$ is an odd element of $W_1(\bar{J}) = Hom(S^2(\bar{J}), \bar{J})$, and $R(\bar{J}, \bar{\mu})$ is a $Str(\bar{J}, \bar{\mu})$-submodule of $W_1(\bar{J})$.

Proposition 3.7 *A superalgebra (J, μ) is rigid if and only if $(\bar{J}, \bar{\mu})$ is rigid.*

Proof. The map $(End J)_- \to (End \bar{J})_-$ defined by: $f \mapsto \bar{f}$, $\bar{f}(\bar{a}) = \overline{f(a)}$, $a \in J$, is an isomorphism of Lie superalgebras sending $Str(J, \mu)$ to $Str(\bar{J}, \bar{\mu})$. Now consider the map $Hom(J \otimes J, J) \to Hom(\bar{J} \otimes \bar{J}, \bar{J})$, $\varphi \mapsto \bar{\varphi}$, defined by: $\bar{\varphi}(\bar{a} \otimes \bar{b}) = (-1)^{p(a)}\overline{\varphi(a \otimes b)}$, $a, b \in J$. This is an isomorphism between the $End(J)_-$-module $Hom(J \otimes J, J)$ and the $End(\bar{J})_-$-module $Hom(\bar{J} \otimes \bar{J}, \bar{J})$, sending $R(J, \mu)$ to $R(\bar{J}, \bar{\mu})$. The statement then follows. □

4 Examples of commutative rigid superalgebras

In the introduction we constructed five infinite-dimensional linearly compact commutative superalgebras: $JS_{1,1}$, $JSHO_{2,2}$, $JSKO_{1,2}$, and $JS_{1,8}^{\alpha}$ ($\alpha = 0, 1$), and a 2-dimensional one, $JS_{0,2}$. In this section we shall construct the remaining four finite-dimensional commutative superalgebras, $JW_{0,4}$, $JW_{0,8}$, $JS_{0,8}$, and $JS_{0,16}$, that appear in the statement of Theorem 0.1. We shall prove in Section 8 that all these superalgebras are indeed simple rigid commutative superalgebras. The proof comes from the generalized TKK construction.

Example 4.1 Let A be a unital commutative associative algebra (i.e. a superalgebra whose underlying superspace is even) with a surjective derivation D. We define on A the following two products:

$$(a) \quad f \circ g = D(f)D(g); \qquad (b) \quad f \circ g = D(fg).$$

(Note that the map D is a central extension homomorphism of (a) to (b).) Then $J = A$, with product (a) or (b), is a rigid commutative algebra. Indeed, in both cases $Str(J, \mu) = AD$ with the usual Lie bracket, acting on J in the obvious way in case (a), and as follows in case (b): $[aD, x] = D(ax)$. Besides, $R(J, \mu) = A\mu$, where, for $f \in A$, $x, y \in J$, $(f\mu)(x, y) = fD(x)D(y)$ in case (a), and $(f\mu)(x, y) = D(fxy)$ in case (b). In both cases $Str(J, \mu)$ acts on $R(J, \mu)$ as follows: $[aD, b\mu] = (aD(b) - 2bD(a))\mu$. It follows that $R(J, \mu) = [Str(J, \mu), \mu]$. Note that, in case (a), $\mathbb{F}1$ is a proper ideal of $(J, \circ)$. A special case of this example is $A = \mathbb{F}[[x]]$ and $D = d/dx =' $ with the product

$f \circ g = f'g'$, called the Beltrami product. This algebra is a central extension of the simple algebra $\mathbb{F}[[x]]$ with the product $f \circ g = (fg)'$, which we called the Beltrami algebra in the introduction. In [6] an interesting quartic identity in these algebras is derived. The Beltrami product is the simplest case of one of the operations introduced by Beltrami in differential geometry.

Remark 4.2 Let $(J, \circ)$ be a non-unital rigid superalgebra. Consider $\tilde{J} = J + \mathbb{F}e$ with the product defined by extending the product of J by: $x \circ e = x$, for $x \in \tilde{J}$. Differently from what happens with usual non unital Jordan superalgebras [4, §5], $\tilde{J}$ is not necessarily a rigid superalgebra. Consider, for example, the central extension J of the Beltrami algebra constructed in Example 4.1, and extend $'$ from J to $\tilde{J}$ by letting $e' = 0$. Then $Str(\tilde{J}, \mu) = \langle L_x, \varphi_x, \mu_e | x \in J\rangle$, where, for $z \in \tilde{J}$, $L_x(z) = x'z'$, and φ_x is defined as follows: $\varphi_x(z) = 0$ for $z \in J$ and $\varphi_x(e) = x$. It follows that $Str(\tilde{J}, \mu).\mu + \mathbb{F}\mu = \langle [L_x, \mu], [\varphi_x, \mu], \mu\rangle$ and one checks that the element $[\varphi_\omega, [\varphi_x, \mu]]$, lying in $R(\tilde{J}, \mu)$, for $\omega, x \in J$, does not belong to $Str(\tilde{J}, \mu).\mu + \mathbb{F}\mu$. It follows that $\tilde{J}$ does not satisfy Definition 3.3.

Given a superspace A, we shall denote, as before, by $\overline{A}$ a copy of A with reversed parity, and given an element $a \in A$, we shall denote the corresponding element of $\overline{A}$ by $\overline{a}$.

Example 4.3 Let A be a commutative associative superalgebra with two odd derivations D_1 and D_2. Let $J(A, D_1, D_2, \mu_1, \mu_2) = AD_1 + AD_2$ and define, for $f, g \in A$, the following product (see the introduction):

$$\begin{gathered} fD_i \circ gD_i = [fD_i, gD_i]_+,\ i = 1, 2, \\ fD_1 \circ gD_2 = [fD_1, gD_2]_+ + (-1)^{p(g)}(\mu_1(f,g)D_1 + \mu_2(f,g)D_2), \end{gathered} \tag{4.1}$$

where μ_1 and μ_2 are some odd type products on A. Then $J(A, D_1, D_2, \mu_1, \mu_2)$ is a simple rigid superalgebra in the following cases:

a) $A = \Lambda(\xi)$, $D_1 = \frac{\partial}{\partial \xi}$ and $D_2 = 0$, $\mu_1(f,g) = fg\xi = \mu_2(f,g)$;

b) $A = \Lambda(\xi_1, \xi_2)$, $D_1 = \frac{\partial}{\partial \xi_1}$, $D_2 = \frac{\partial}{\partial \xi_2} + \xi_1\xi_2\frac{\partial}{\partial \xi_1}$, $\mu_1(f,g) = fg\xi_1 = \mu_2(f,g)$;

c) $A = \Lambda(\xi_1, \xi_2)$, $D_1 = \frac{\partial}{\partial \xi_1} + \xi_1\xi_2\frac{\partial}{\partial \xi_2}$, $D_2 = \frac{\partial}{\partial \xi_2} + \xi_1\xi_2\frac{\partial}{\partial \xi_1} - \xi_1\xi_2\frac{\partial}{\partial \xi_2}$, $\mu_1(f,g) = fg(\xi_1 + \xi_2)$, $\mu_2(f,g) = fg\xi_1$;

d) $A = \Lambda(\xi_1, \xi_2, \xi_3)$, $D_1 = \frac{\partial}{\partial \xi_1} + \xi_1\xi_2\frac{\partial}{\partial \xi_3}$, $D_2 = \frac{\partial}{\partial \xi_2} + \xi_1\xi_2\frac{\partial}{\partial \xi_1} + \xi_2\xi_3\frac{\partial}{\partial \xi_3} + \xi_1\xi_2\frac{\partial}{\partial \xi_3}$, $\mu_1 = 0 = \mu_2$;

We shall denote the superalgebras listed in a)-d) by $JW_{0,4}$, $JW_{0,8}$, $JS_{0,8}$, and $JS_{0,16}$, respectively.

Remark 4.4 For every $\alpha \neq 0$, product (0.12) is isomorphic to (4.1) where $D_1 = \frac{\partial}{\partial \xi_1} + \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial x}$, $D_2 = \frac{\partial}{\partial \xi_2}$ and $\mu_1 = 0 = \mu_2$.

In Section 9 we shall prove the following theorem.

Theorem 4.5 *Any simple linearly compact commutative rigid superalgebra, which is not a Jordan superalgebra, is isomorphic either to one of the five infinite-dimensional superalgebras* $JS_{1,1}$, $JSHO_{2,2}$, $JSKO_{1,2}$, $JS^{\alpha}_{1,8}$ *for* $\alpha = 0, 1$, *or to one of the five finite-dimensional superalgebras* $JS_{0,2}$, $JW_{0,4}$, $JW_{0,8}$, $JS_{0,8}$, $JS_{0,16}$.

5 Examples of anti-commutative rigid superalgebras

We shall prove in Section 8 that the series $LP(n,n)$ and $LP(n,n+1)$ from Example 5.1, as well as examples 5.4–5.11, 5.13–5.15, 5.17–5.18, 5.20, 5.22–5.24 in this section are indeed simple rigid anti-commutative superalgebras. The proof comes from the generalized TKK construction. With the exception of the series $LP(n,n)$ and $LP(n,n+1)$, we shall denote the anti-commutative simple rigid superalgebras by $LX_{m,n}$, where the first index is the growth of the superalgebra and the second is its total size (equal to the total dimension if the size is 0). The letter X comes from the connection to the Lie superalgebra $X(m',n')$ via the generalized TKK construction.

Example 5.1 Let P be an odd generalized Poisson superalgebra with the bracket $\{\cdot,\cdot\}$ and the derivation D (defined by $D(a)=\{e,a\}$), let $'=d/dx$ on $P[[x]]$ and let η be an odd indeterminate, such that $\eta^2=0$. Consider the commutative associative superalgebra $J=P[[x]]+\eta P[[x]]$ and extend on J, by commutativity, the following odd product ($f,g\in P,\ a,b\in\mathbb{F}[[x]]$):

$$\begin{aligned} &fa\circ gb=(-1)^{p(f)+1}\{f,g\}ab+\tfrac{1}{2}(-1)^{p(f)}fD(g)xa'b+\tfrac{1}{2}D(f)gxb'a+2\eta fgab\\ (5.1)\qquad &fa\circ\eta(gb)=\eta\{f,g\}ab-((-1)^{p(f)}fg+\tfrac{1}{2}\eta fD(g)x)a'b-\tfrac{1}{2}(-1)^{p(f)}\eta D(f)ga(b+xb')\\ &\eta(fa)\circ\eta(gb)=(-1)^{p(f)}\eta fg(a'b-ab'). \end{aligned}$$

We will denote the resulting odd type superalgebra by OJP. If $P=PO(n,n)$ (resp. $P=PO(n,n+1)$), then we will denote it by $OJP(n,n)$ (resp. $OJP(n,n+1)$).

Now we shall show that OJP is a rigid odd type superalgebra. Denote by μ_v the operator of left multiplication by $v\in J$ with respect to the product $\circ$, defined by (5.1), and by l_v the operator of left multiplication by v with respect to the associative product on J. Then $Str(J)=\langle\mu_v,l_v\mid v\in J\rangle$. Indeed, for $v,w\in J$, we have:

$$[\mu_v,\mu_w]=(-1)^{p(v)+1}(\mu_{v\circ w-2\eta vw}+2l_{\eta(v\circ w)+(vw)'+2\eta D(vw)}).$$

If we take $v=fa$ and $w=\eta gb$, for $f,g\in P$ and $a,b\in\mathbb{F}[[x]]$, then $\eta(v\circ w)+(vw)'+2\eta D(vw)=\eta fgab'$, therefore $Str(J)$ contains all l_z for $z\in\eta P[[x]]$. If we take $v=fa$ and $w=gb$, then $(vw)'=fg(ab)'$, hence $Str(J)$ contains also all l_z for $z\in P[[x]]$. Finally we have:

$$[\mu_v,l_w]=l_{v\circ w-2\eta vw}+l_{D(v)w},\ \ [l_v,l_w]=0.$$

Next, the following commutation relations hold:

$$(5.2)\qquad [l_v,[l_w,\mu]]=-[l_{vw},\mu]+(-1)^{p(v)}[\mu_{z-2\eta\psi}-2l_{\eta z},\mu]$$

for some $z,\psi\in J$ such that $z'=(-1)^{p(v)+1}(v\circ w-2\eta vw+D(vw))$, $\psi'=D(z)$;

$$(5.3)\qquad [l_v,[\mu_w,\mu]]=(-1)^{p(v)+1}[\mu_{vw},\mu]+2(-1)^{p(v)+1}[\mu_{z-2\eta\psi}-2l_{\eta z},\mu],$$

for some $z,\psi\in J$ such that $z'=v'w+2\eta D(vw)$, $\psi'=D(z)$;

$$(5.4)\qquad [\mu_w,[l_v,\mu]]=[l_{w\circ v-2\eta wv},\mu]+[l_{D(w)v},\mu]+(-1)^{p(v)(p(w)+1)}[l_v,[\mu_w,\mu]];$$

$$(5.5) \qquad [\mu_v,[\mu_w,\mu]]=2[l_{v\circ\eta w},\mu]+(-1)^{p(v)+1}([\mu_{D(v)w},\mu]-2[\mu_{\eta z},\mu]+[\mu_{\varphi-2\eta\psi}-2l_{\eta\varphi},\mu])$$

for some $z,\varphi,\psi\in J$ such that $z'=vw'-v'w-\{v,D(w)\}+2(-1)^{p(v)}D(v)D(w)$, $\varphi'=2\eta vw'-v\circ w'-D(v)w'-2(-1)^{p(v)+1}v'D(w)-D(v')w$, $\psi'=D(\varphi)$. Relations (5.2)–(5.5) show that $[Str(J,\mu),\mu]+\mathbb{F}\mu=R(J,\mu)$, hence OJP is a rigid odd type superalgebra. Notice that relations (5.2)–(5.5) do not depend on the choice of the functions z and ψ, since, for every $c\in\mathbb{F}$, $\mu_c=2l_{\eta c}-l_c\circ D$, as it can be deduced from (5.1), $[l_c\circ D,\mu]=0$, and $[\mu_{\eta c},\mu]=0$.

Let $LP=\overline{OJP}$ with product $\overline{f}\bullet\overline{g}=(-1)^{p(f)}\overline{f\circ g}$. By Proposition 3.7, LP is an anti-commutative rigid superalgebra. If $P=PO(n,n)$ (resp. $P=PO(n,n+1)$), then we will denote the corresponding anti-commutative rigid superalgebra by $LP(n,n)$ (resp. $LP(n,n+1)$).

Remark 5.2 The generalized odd Poisson bracket on P can be recovered from product (5.1) as follows ($f,g\in P$): $\{f,g\}=(-1)^{p(f)+1}f\circ g+2\eta(fx)\circ\eta g$. Furthermore, $(-1)^{p(f)}\eta fx\circ\eta g=\eta fg$, hence also the associative product on P can be recovered from product (5.1).

Remark 5.3 The odd Poisson superalgebra $PO(n+1,n+1)$ can be decomposed as follows: $PO(n+1,n+1)=PO(n,n)[[x_{n+1}]]+\xi_{n+1}PO(n,n)[[x_{n+1}]]$. Therefore, if $P=PO(n,n)$, then $LP(n,n)=PO(n+1,n+1)$ with reversed parity and the following product:

$$f\bullet g=-\{f,g\}_H+2(-1)^{p(f)}\xi_{n+1}fg$$

where $\{\cdot,\cdot\}_H$ is the Buttin bracket on $PO(n+1,n+1)$.

Likewise, $PO(n+1,n+2)=PO(n,n+1)[[x_{n+1}]]+\xi_{n+1}PO(n,n+1)[[x_{n+1}]]$ and if $P=PO(n,n+1)$, then $LP(n,n+1)=PO(n+1,n+2)$ with reversed parity and product

$$f\bullet g=-\{f,g\}_K+2(-1)^{p(f)}\xi_{n+1}fg$$

where $\{\cdot,\cdot\}_K$ is the odd generalized Poisson bracket on $PO(n+1,n+2)$.

Example 5.4 Let us define on $\mathcal{O}=\mathcal{O}(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$, with $n\geq 2$, the following product ($f,g\in\mathcal{O}$):

$$(5.6) \qquad f\bullet g=\{f,g\}+2(-1)^{p(f)+1}\xi_1fg+2\{x_2\xi_1\xi_2f,g\}+2(-1)^{p(f)}\{x_2\xi_1\xi_2,f\}g$$

where $\{\cdot,\cdot\}$ is the usual odd Poisson bracket on $\mathcal{O}(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$.

Let $J'=\{f\in\mathcal{O}\mid\Delta(f)=0\}$, where Δ is the odd Laplacian, i.e., $\Delta=\sum_{i=1}^n\frac{\partial^2}{\partial x_i\partial\xi_i}$. Then J' is closed under product (5.6). Besides, the span J of all monomials in J' except for $\xi_1\dots\xi_n$ is an ideal of $(J',\bullet)$.

J with reversed parity and product (5.6) is a simple rigid anti-commutative superalgebra that we will denote by $LSHO_{n,2^{n-1}}$.

Example 5.5 Let us define on $\mathcal{O}=\mathcal{O}(x_1,\dots,x_n,\xi_1,\dots,\xi_n,\tau)$, for $n\geq 1$, the following product ($f,g\in\mathcal{O}$):

$$(5.7) \quad f\bullet g=\{f,g\}_K+\{\xi_1\tau f,g\}_K+(-1)^{p(f)+1}(\xi_1(2E-(n+1)\beta)(f)g+\{\xi_1\tau,f\}_Kg-2\tau\frac{\partial f}{\partial x_1}g)$$

where $\{\cdot,\cdot\}_K$ is the usual generalized odd Poisson bracket on $\mathcal{O}(x_1,\dots,x_n,\xi_1,\dots,\xi_n,\tau)$, E is the Euler operator, and $\beta\in\mathbb{F}$. Let $J'=\{f\in\mathcal{O}\mid div_\beta(f)+(1-\beta)\frac{\partial f}{\partial\tau}=0\}$, where div_β is the

β-divergence introduced in Example 2.3. Then J' is closed under product (5.7). Besides, if $\beta = 1$ (resp. $\beta = \frac{n-1}{n+1}$), then the span J_β of all monomials in J' except for $\xi_1 \dots \xi_n \tau$ (resp. $\xi_1 \dots \xi_n$) is an ideal of $(J', \bullet)$. For $\beta \neq 1, \frac{n-1}{n+1}$, let us set $J_\beta = J'$. Then, for every $\beta \neq \frac{4}{n+1}$, J_β with reversed parity and product $\bullet$ is a simple rigid anti-commutative superalgebra that we shall denote by $LSKO_{n,2^n}(\beta)$.

Example 5.6 Let $n = 2$, $\beta = 1$ and J_β be as in Example 5.5. Consider the following product on J_1:

$$f \bullet g = \{f, g\}_K + \{\xi_2(\tau - \xi_1)f, g\}_K - (-1)^{p(f)+1}(\xi_2 fg - \{\xi_2(\tau - \xi_1), f\}g - \frac{\partial f}{\partial \tau}\xi_1\xi_2 g).$$

Then J_1 with reversed parity and product $\bullet$ is a simple rigid anti-commutative superalgebra that we will denote by $LSKO'_{2,4}$.

Example 5.7 Let $LW_{0,2} = \mathbb{F}a + \mathbb{F}\bar{a}$, where a is even and $\bar{a}$ is odd, with product

$$a \bullet a = 0, \quad a \bullet \bar{a} = -\bar{a} \bullet a = \bar{a}, \quad \bar{a} \bullet \bar{a} = a.$$

Example 5.8 Let $LW_{1,2} = \mathbb{F}[[x]] + \overline{\mathbb{F}[[x]]}$ with product

$$f \bullet g = 0, \quad f \bullet \bar{g} = -\bar{g} \bullet f = \overline{fg}, \quad \bar{f} \bullet \bar{g} = 2fg - \frac{d(fg)}{dx}.$$

Example 5.9 Let $LHO_{1,2} = \mathbb{F}[[x]]/\mathbb{F}1 + \overline{\mathbb{F}[[x]]}$ with product $(f, g \in \mathbb{F}[[x]])$:

$$f \bullet g = 0, \quad f \bullet \overline{g} = -\overline{g} \bullet f = \overline{\frac{df}{dx}g}, \quad \overline{f} \bullet \overline{g} = 2fg. \tag{5.8}$$

Example 5.10 Let $LSHO'_{2,2} = \mathbb{F}[[x_1, x_2]]/\mathbb{F}1 + \overline{\mathbb{F}[[x_1, x_2]]}$ with product $(f, g \in \mathbb{F}[[x_1, x_2]])$:

$$f \bullet g = -\{f, g\}, \quad f \bullet \bar{g} = -\bar{g} \bullet f = \overline{\{f, g\}} + \overline{gD_2(f)}, \quad \bar{f} \bullet \bar{g} = -2fg$$

where $\{f, g\} = D_1(f)D_2(g) - D_2(f)D_1(g)$, $D_1 = (1 + x_1)\frac{\partial}{\partial x_1}$ and $D_2 = \frac{\partial}{\partial x_2}$.

Example 5.11 Let $LW^{\alpha}_{1,2} = \mathbb{F}[[x]]\frac{d}{dx} + \mathbb{F}[[x]]$ with product $(f, g \in \mathbb{F}[[x]],\ \alpha \in \mathbb{F})$:

$$f\frac{d}{dx} \bullet g\frac{d}{dx} = \left[f\frac{d}{dx}, g\frac{d}{dx}\right]_-, \quad f\frac{d}{dx} \bullet g = -g \bullet f\frac{d}{dx} = -(\alpha + x)(fg)\frac{d}{dx} + f\frac{dg}{dx}, \quad f \bullet g = 0. \tag{5.9}$$

Remark 5.12 For every $\alpha \neq 0$, product (5.9) is isomorphic to the following $(f, g \in \mathbb{F}[[x]],\ \alpha \in \mathbb{F})$:

$$f\frac{d}{dx} \bullet g\frac{d}{dx} = \left[f\frac{d}{dx}, g\frac{d}{dx}\right]_-, \quad f\frac{d}{dx} \bullet g = -g \bullet f\frac{d}{dx} = -(fg)\frac{d}{dx} + f\frac{dg}{dx}, \quad f \bullet g = 0. \tag{5.10}$$

Example 5.13 Let $LS_{1,3} = \mathbb{F}[[x]]\frac{d}{dx} + \mathbb{F}[[x]] + \widetilde{\mathbb{F}[[x]]}$ with the following product $(f, g \in \mathbb{F}[[x]])$:

$$f\frac{d}{dx} \bullet g\frac{d}{dx} = \left[f\frac{d}{dx}, g\frac{d}{dx}\right]_-, \quad f \bullet g = 0, \quad \tilde{f} \bullet \tilde{g} = 0,$$

$$f\frac{d}{dx} \bullet g = f\frac{dg}{dx}, \quad f\frac{d}{dx} \bullet \tilde{g} = \widetilde{f\frac{dg}{dx}}, \quad f \bullet \tilde{g} = fg\frac{d}{dx},$$

extended to all other pairs of elements of $LS_{1,3}$ by anti-commutativity.

Example 5.14 Let $LW_{2,2}^{\alpha} = \mathbb{F}[[x_1,x_2]]\frac{\partial}{\partial x_1} + \mathbb{F}[[x_1,x_2]]\frac{\partial}{\partial x_2}$ with the following product ($f, g \in \mathbb{F}[[x_1,x_2]]$, $\alpha \in \mathbb{F}$):

$$\begin{aligned} f\frac{\partial}{\partial x_i} \bullet g\frac{\partial}{\partial x_i} &= \left[f\frac{\partial}{\partial x_i}, g\frac{\partial}{\partial x_i}\right]_-, \quad i=1,2 \\ f\frac{\partial}{\partial x_1} \bullet g\frac{\partial}{\partial x_2} = -g\frac{\partial}{\partial x_2} \bullet f\frac{\partial}{\partial x_1} &= [f\frac{\partial}{\partial x_1}, g\frac{\partial}{\partial x_2}]_- + fg(1+x_1)\frac{\partial}{\partial x_1} - \alpha x_2 fg\frac{\partial}{\partial x_2}. \end{aligned} \tag{5.11}$$

Example 5.15 Let $LS_{2,2}^{\alpha} = \mathbb{F}[[x_1,x_2]]D_1 + \mathbb{F}[[x_1,x_2]]D_2$, where $D_1 = \frac{\partial}{\partial x_1}$ and $D_2 = \frac{\partial}{\partial x_2} + \alpha x_1\frac{\partial}{\partial x_2} + x_1x_2\frac{\partial}{\partial x_2}$, $\alpha \in \mathbb{F}$, with the following product ($f, g \in \mathbb{F}[[x_1,x_2]]$):

$$\begin{aligned} fD_i \bullet gD_i &= [fD_i, gD_i]_-, \quad i=1,2 \\ fD_1 \bullet gD_2 = -gD_2 \bullet fD_1 &= [fD_1, gD_2]_- + x_1fgD_1. \end{aligned} \tag{5.12}$$

Remark 5.16 For every $\alpha \neq 0$, $LS_{2,2}^{\alpha}$ is isomorphic to the anti-commutative superalgebra $J' = \mathbb{F}[[x_1,x_2]]D_1' + \mathbb{F}[[x_1,x_2]]D_2'$, where $D_1' = D_1$, $D_2' = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_2}$ and the product is defined as follows ($f, g \in \mathbb{F}[[x_1,x_2]]$):

$$\begin{aligned} fD_i' \bullet gD_i' &= [fD_i', gD_i']_-, \quad i=1,2 \\ fD_1' \bullet gD_2' = -gD_2' \bullet fD_1' &= [fD_1', gD_2']_-. \end{aligned} \tag{5.13}$$

Example 5.17 Let $J = \mathbb{F}[[x]]\frac{d}{dx} + \overline{\mathbb{F}[[x]]}$. Consider the following product ($f, g \in \mathbb{F}[[x]]$, $\beta \in \mathbb{F}$):

$$f\frac{d}{dx} \bullet g\frac{d}{dx} = -\beta\left[f\frac{d}{dx}, g\frac{d}{dx}\right]_-, \quad f\frac{d}{dx} \bullet \overline{g} = -\overline{g} \bullet f\frac{d}{dx} = -\overline{g\frac{df}{dx}} + \beta\overline{f\frac{dg}{dx}}, \quad \overline{f} \bullet \overline{g} = xfg\frac{d}{dx}.$$

Then, for every $\beta \neq 0, 1, 2+\frac{1}{b}, 2+\frac{2}{b}$, $b \in \mathbb{Z}_{>0}$, $(J, \bullet)$ is a simple rigid anti-commutative superalgebra that we shall denote by $LSKO'_{1,2}(\beta)$.

Example 5.18 Let $LH_{2,2}^{\alpha} = \mathbb{F}[[x]]\frac{d}{dx} + \overline{\mathbb{F}[[x]]}/\mathbb{F}\bar{1}$ with the following product ($f, g \in \mathbb{F}[[x]]$, $\alpha \in \mathbb{F}$):

$$f\frac{d}{dx} \bullet g\frac{d}{dx} = \left[f\frac{d}{dx}, g\frac{d}{dx}\right]_-, \quad f\frac{d}{dx} \bullet \bar{g} = -\bar{g} \bullet f\frac{d}{dx} = \overline{f\frac{dg}{dx}}, \quad \bar{f} \bullet \bar{g} = -2(\alpha+x)\frac{df}{dx}\frac{dg}{dx}. \tag{5.14}$$

Remark 5.19 For every $\alpha \neq 0$, product (5.14) is isomorphic to the following ($f, g \in \mathbb{F}[[x]]$, $\alpha \in \mathbb{F}$):

$$f\frac{d}{dx} \bullet g\frac{d}{dx} = \left[f\frac{d}{dx}, g\frac{d}{dx}\right]_-, \quad f\frac{d}{dx} \bullet \bar{g} = -\bar{g} \bullet f\frac{d}{dx} = \overline{f\frac{dg}{dx}}, \quad \bar{f} \bullet \bar{g} = -2\frac{df}{dx}\frac{dg}{dx}.$$

Let A be a commutative associative unital algebra. We call a generalized bivector field any finite sum of the form $Z + \sum_i X_i \wedge Y_i$, where Z, X_i, Y_i are derivations of A. The corresponding quasi Poisson bracket on A is:

$$\{f, g\} = Z(f)g - fZ(g) + \sum_i X_i(f)Y_i(g) - Y_i(f)X_i(g).$$

This is a skew-symmetric product which in general does not satisfy the Jacobi identity. The following examples of anti-commutative rigid algebras come from quasi Poisson brackets.

Example 5.20 $LHO^{\alpha}_{3,1} = \mathbb{F}[[x_1, x_2, x_3]]/\mathbb{F}1$ with the quasi Poisson bracket associated with the bivector field $\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + (\alpha x_1 + x_1^2)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3}$, $\alpha \in \mathbb{F}$. We call this the Jacobi-Mayer algebra.

Remark 5.21 If $\alpha \neq 0$ in Example 5.20, we can replace $\alpha x_1 + x_1^2$ by x_1, getting an isomorphic algebra. This algebra appears in the theory of PDEs under the name Jacobi-Mayer bracket (see [7]).

Example 5.22 $LSHO^{\alpha}_{4,1} = \mathbb{F}[[x_1, x_2, x_3, x_4]]/\mathbb{F}1$ with the quasi Poisson bracket associated with the bivector field $\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + (\alpha + x_1)\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$, $\alpha \in \mathbb{F}$.

Example 5.23 $LKO_{2,1} = \mathbb{F}[[x_1, x_2]]$ with the quasi Poisson bracket associated with the generalized bivector field $2\frac{\partial}{\partial x_1} + (x_1 + x_2)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$.

Example 5.24 Let $LSKO^{\alpha}_{3,1}(\beta) = \mathbb{F}[[x_1, x_2, x_3]]$ with the quasi Poisson bracket associated with the generalized bivector field $\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - 3\beta(\alpha + x_1)x_1\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} - (\alpha + 2x_1)(x_2\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} - 2\frac{\partial}{\partial x_3})$, where $\alpha, \beta \in \mathbb{F}$. Then, for every $\alpha \neq 0$ and $\beta \neq \frac{1}{3}$ and for $\alpha = 0$, $LSKO^{\alpha}_{3,1}(\beta)$ is a simple rigid anti-commutative superalgebra.

In Section 9 we shall prove the following theorem.

Theorem 5.25 *Any simple linearly compact anti-commutative rigid superalgebra, which is not a Lie superalgebra, is isomorphic to one of the following superalgebras:*

- $LP(n, n)$ *or* $LP(n, n + 1)$ *for* $n \geq 0$*;*
- $LSKO'_{2,4}$, $LSHO_{n,2^{n-1}}$ *or* $LSKO_{n-1,2^{n-1}}(\beta)$ *for* $n \geq 2$ *and for* $\beta \neq \frac{4}{n}$*;*
- $LW_{k,2}$ *for* $k = 0, 1$, $LHO_{1,2}$, $LSHO'_{2,2}$*;*
- $LW^{\alpha}_{1,2}$ *for* $\alpha = 0, 1$*;*
- $LW^{\alpha}_{2,2}$, $LS^{\alpha}_{2,2}$ *for* $\alpha = 0, 1$*;*
- $LS_{1,3}$*;*
- $LH^{\alpha}_{1,2}$ *for* $\alpha = 0, 1$, $LSKO'_{1,2}(\beta)$ *for* $\beta \neq 0, 1, 2 + \frac{1}{b}, 2 + \frac{2}{b}, b \in \mathbb{Z}_{>0}$*;*
- $LKO_{2,1}$, $LHO^{\alpha}_{3,1}$, $LSHO^{\alpha}_{4,1}$, $LSKO^{\alpha}_{3,1}(\beta)$ *for* $\alpha = 0$ *or* $\alpha = 1$ *and* $\beta \neq \frac{1}{3}$.

6 Admissible gradings

Definition 6.1 *An even admissible (resp. odd admissible) grading of a Lie superalgebra* $\mathfrak{g}$ *is a transitive* $\mathbb{Z}$*-grading* $\mathfrak{g} = \prod_{j \geq -1} \mathfrak{g}_j$ *such that there exists an even (resp. odd) element* $\mu \in \mathfrak{g}_1$ *satisfying the following properties:*

(6.1) $$[\mathfrak{g}_0, \mu] + \mathbb{F}\mu = \mathfrak{g}_1;$$

(6.2) *the subspace* $[\mathfrak{g}_{-1}, \mu]$ *generates the Lie superalgebra* $\mathfrak{g}_0$*;*

(6.3) $$\mathfrak{g}_k = \mathfrak{g}_1^k \text{ for every } k \geq 1.$$

In [3, §1] the notions of growth and size of an artinian linearly compact Lie superalgebra $\mathfrak{g}$, related to a Weisfeiler filtration of $\mathfrak{g}$, were defined. We define the growth and size of a subspace of $\mathfrak{g}$ as those for the induced filtration from a Weisfeiler filtration of $\mathfrak{g}$. This definition is independent of the choice of the filtration due to the equivalence of Weisfeiler filtrations for any artinian $\mathfrak{g}$.

Remark 6.2 If $\mathfrak{g} = \prod_{j\geq -1}\mathfrak{g}_j$ is an even admissible or odd admissible grading of a Lie superalgebra $\mathfrak{g}$ with $\dim\mathfrak{g}_0 < \infty$, then, by condition (6.1), $\dim\mathfrak{g}_1 \leq \dim\mathfrak{g}_0 + 1$. If $\mathfrak{g}_0$ is infinite-dimensional and $\mathfrak{g}$ is artinian linearly compact, then, by condition (6.1), growth $\mathfrak{g}_1 \leq$ growth $\mathfrak{g}_0$ and size $\mathfrak{g}_1 \leq$ size $\mathfrak{g}_0$.

Proposition 6.3 *Let $\mathfrak{g} = \prod_{j\geq -1}\mathfrak{g}_j$ be an even admissible (resp. odd admissible) grading of a Lie superalgebra $\mathfrak{g}$, and let $\mu \in (\mathfrak{g}_1)_{\bar 0}$ (resp. $\mu \in (\mathfrak{g}_1)_{\bar 1}$) be as in Definition 6.1. Define the following product on $J = \mathfrak{g}_{-1}$:*

$$x \circ y = [[\mu, x], y] \qquad (x, y \in J). \tag{6.4}$$

Then $(J, \circ)$ is a rigid (resp. rigid odd type) superalgebra that we will denote by $J(\mathfrak{g}, \mu)$.

Proof. By definition, for $x \in \mathfrak{g}_{-1}$, $\mu_x = ad\,[\mu, x]$, hence, by property (6.2), $Str(J, \mu) = \mathfrak{g}_0$ acting on $J = \mathfrak{g}_{-1}$ via the adjoint action. Therefore, by property (6.1), $[Str(J, \mu), \mu] + \mathbb{F}\mu = \mathfrak{g}_1$, hence $R(J, \mu) = \mathfrak{g}_1$ and Definition 3.3 is satisfied. $\square$

Remark 6.4 Let $\mathfrak{g} = \prod_{i\geq -1}\mathfrak{g}_i$ be an even admissible grading with $\mu \in (\mathfrak{g}_1)_{\bar 0}$ as in Definition 6.1. The grading is *short*, i.e. $\mathfrak{g}_i = 0$ for $i > 1$, if and only if $J(\mathfrak{g}, \mu)$ is a Jordan superalgebra [4, Proposition 5.1, Lemma 5.13].

Proposition 6.5 *Let $\mathfrak{g} = \prod_{j\geq -1}\mathfrak{g}_j$ be an odd admissible grading of a Lie superalgebra $\mathfrak{g}$, and let $\mu \in (\mathfrak{g}_1)_{\bar 1}$ be an element as in Definition 6.1, such that $[\mu, \mu] = 0$. Then $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \cong \xi\mathfrak{g}_0 + \mathfrak{g}_0 + \mathbb{F}\frac{d}{d\xi}$, for some odd indeterminate ξ.*

Proof. Since $[\mu, \mu] = 0$, by the Jacobi identity one has $(ad\,\mu)^2 = 0$. Now consider product (6.4) on $\mathfrak{g}_{-1}$. Condition $(ad\,\mu)^2 = 0$ implies the following identity

$$(a \circ b) \circ c = (-1)^{p(a)+1} a \circ (b \circ c) - (-1)^{p(b)(p(a)+1)} b \circ (a \circ c).$$

It follows that, if we set $[a, b]' = (-1)^{p(a)} a \circ b$, then $[\cdot, \cdot]'$ defines on $\mathfrak{g}_{-1}$ with reversed parity a Lie superalgebra bracket. By condition (6.2), $\mathfrak{g}_0$ is generated by the left translations $\mu_x = [\mu, x]$ with $x \in \mathfrak{g}_{-1}$, and the map $x \mapsto -\mu_x$, from $(\mathfrak{g}_{-1}, [\cdot, \cdot]')$ with reversed parity to $\mathfrak{g}_0$, defines an isomorphism of Lie superalgebras since $(ad\,\mu)^2 = 0$. Besides, $(ad\,\mu)^2 = 0$ implies $[\mu, \mu_x] = 0$ for every $x \in \mathfrak{g}_{-1}$, hence, by condition (6.1), $\mathfrak{g}_1 = \mathbb{F}\mu$. $\square$

Remark 6.6 Let $\mathfrak{g} = \oplus_{j=-1}^{d}\mathfrak{g}_j$ be an even admissible grading of a Lie superalgebra $\mathfrak{g}$, with $\mu \in (\mathfrak{g}_1)_{\bar 0}$ as in Definition 6.1. Suppose that $J = \mathfrak{g}_{-1}$ with product (6.4) is a rigid superalgebra with unit element e. Set $h = \mu_e$, then $\{e, h, \mu\}$ is an sl_2 triple, whose adjoint action on $\mathfrak{g}$ exponentiates to the action of SL_2 by continuous automorphisms on $\mathfrak{g}$, since the grading on $\mathfrak{g}$ has finite height. Therefore we have an automorphism of $\mathfrak{g}$ which exchanges $\mathfrak{g}_{-1}$ with $\mathfrak{g}_1$, hence the grading is short, i.e., $\mathfrak{g}_i = 0$ for $i > 1$, and therefore J is a Jordan superalgebra.

7 Generalized TKK construction

Definition 7.1 *Let J be a commutative superalgebra with product μ (even or odd). We denote by $Lie(J,\mu)$ the $\mathbb{Z}$-graded Lie subalgebra of $W(J)$ generated by $J = W_{-1}(J)$ and μ.*

Proposition 7.2 *Let J be a commutative superalgebra with product μ. Then:*

(a) *The $\mathbb{Z}$-grading of $W(J)$ induces on $Lie(J,\mu)$ a $\mathbb{Z}$-grading $Lie(J,\mu) = \prod_{k\geq -1} Lie_k(J)$, where $Lie_{-1}(J) = J$, $Lie_0(J) = Str(J,\mu)$ and $Lie_1(J) = R(J,\mu)$, which is admissible if J is rigid.*

(b) *J is simple if and only if $Lie(J,\mu)$ is an irreducible $\mathbb{Z}$-graded Lie superalgebra;*

(c) *If J is linearly compact then $Lie(J,\mu)$ is linearly compact.*

Proof. The transitivity of the grading and properties (6.2) and (6.3) follow from the definitions. Moreover, if J is a rigid superalgebra, then also property (6.1) holds, i.e., the $\mathbb{Z}$-grading defined on $Lie(J,\mu)$ is admissible, proving (a). Denote by μ the commutative product of J. By construction, if x,y lie in J, then $x \circ y = [[\mu,x],y]$, hence a proper ideal of J is a proper $Str(J,\mu)$-submodule of $Lie_{-1}(J)$, proving (b). Finally, (c) follows from (a) and [4, Lemma 2.1]. $\square$

Theorem 7.3 *(A) If J is a simple linearly compact rigid superalgebra with an even product μ, then one of the following two possibilities holds:*

(a) *$Lie(J,\mu)$ is a simple linearly compact Lie superalgebra;*

(b) *$Lie(J,\mu) = S \rtimes \mathbb{F}\mu$, where S is a simple linearly compact Lie superalgebra and μ is an even outer derivation of S.*

(A′) If J is a simple linearly compact rigid odd type superalgebra with an odd product μ, then one of the following four possibilities holds:

(a′) *$Lie(J,\mu)$ is a simple linearly compact Lie superalgebra;*

(b′) *$Lie(J,\mu) = S + \mathbb{F}\mu + \mathbb{F}[\mu,\mu]$, where S is a simple linearly compact Lie superalgebra and μ is an odd outer derivation of S such that $[\mu,\mu] \neq 0$;*

(c′) *$Lie(J,\mu) \cong \xi\mathfrak{a} + \mathfrak{a} + d/d\xi$, where $\mathfrak{a}$ is a simple linearly compact Lie superalgebra and ξ is an odd indeterminate;*

(d′) *$Lie(J,\mu) = S \otimes \Lambda(1) + \mathbb{F}\mu + \mathbb{F}d$ where S is a simple linearly compact Lie superalgebra, d is an even outer derivation of S and $\mu = d \otimes \xi + 1 \otimes d/d\xi$.*

Proof. By Proposition 7.2, $Lie(J,\mu)$ is a transitive irreducible $\mathbb{Z}$-graded Lie superalgebra. Let I' be a non-zero closed ideal of $Lie(J,\mu)$. Then, by transitivity, $I' \cap Lie_{-1}(J,\mu) \neq \emptyset$, hence, by the irreducibility of the grading, $I' \cap Lie_{-1}(J,\mu) = J$. Now, let I be the intersection of all non-zero closed ideals of $Lie(J,\mu)$. Then, by the above remark, I is a minimal closed ideal of $Lie(J,\mu)$, containing J. therefore, by construction, $Lie(J,\mu) = I + \mathbb{F}\mu + \mathbb{F}[\mu,\mu]$. Next, by the super-analogue of Cartan-Guillemin's theorem [2, 9], established in [8], $I = S\hat{\otimes}\Lambda(m,n)$, for some simple linearly compact Lie superalgebra S and some $m,n \in \mathbb{Z}_{\geq 0}$, and μ lies in $Der(S\hat{\otimes}\mathcal{O}(m,n))$.

Since $Der(S\hat{\otimes}\mathcal{O}(m,n)) = DerS\hat{\otimes}\mathcal{O}(m,n) + 1\otimes W(m,n)$ [8], we have: $\mu = \sum_i (d_i \otimes a_i) + 1\otimes \mu'$ for some $d_i \in DerS$, $a_i \in \mathcal{O}(m,n)$ and $\mu' \in W(m,n)$.

First consider the case when J is a simple rigid superalgebra. Then μ' is an even element of $W(m,n)$ hence, by the minimality of the ideal $S\hat{\otimes}\mathcal{O}(m,n)$, $n=0$. Now suppose $m \geq 1$. If μ' lies in the non-negative part of $W(m,0)$ with the principal grading, then the ideal generated by Sx_1 is a proper μ-invariant ideal of $S\hat{\otimes}\mathcal{O}(m,0)$, contradicting its minimality. Therefore we may assume, up to a linear change of indeterminates, that $\mu' = \frac{\partial}{\partial x_1} + D$, for some derivation D lying in the non-negative part of $W(m,0)$. Since μ lies in $Lie_1(J)$, we have $\deg(x_1) = -1$, but this is a contradiction since the $\mathbb{Z}$-grading of $Lie(J,\mu)$ has depth 1. It follows that $m=0$. Therefore, either μ is an inner derivation of S and $Lie(J,\mu) = S$, or μ is an outer derivation of S and $Lie(J,\mu) = S \rtimes \mathbb{F}\mu$ ($[\mu,\mu]=0$ since μ is even).

Now consider the case when J is a simple rigid odd type superalgebra. Consider the principal grading of $W(m,n)$ and denote by $W(m,n)_{\geq 0}$ its non-negative part. If $\mu' \in W(m,n)_{\geq 0}$, then the minimality of the ideal $S\hat{\otimes}\mathcal{O}(m,n)$ implies $m=n=0$, hence we get (a'), in case μ is an inner derivation of S, or we get (b'), in case μ is an outer derivation of S such that $[\mu,\mu] \neq 0$, or we get (c') by Proposition 6.5, in case μ is an outer derivation of S such that $[\mu,\mu]=0$. Now let $n \geq 1$ and suppose that μ' has a non-zero projection on $W(m,n)_{-1}$. Then, up to a linear change of indeterminates, $\mu' = \frac{\partial}{\partial \xi_1} + D$ for some odd derivation $D \in W(m,n)_{\geq 0}$. Write $D = \xi_1 D_0 + D_1$ where $D_0 \in W(m,n)_{\bar{0}} \cap (\mathcal{O}(x_1,\dots,x_m,\xi_2,\dots,\xi_n)\otimes W(m,n)_{-1})$ and $D_1 \in W(m,n)_{\bar{1}} \cap (\mathcal{O}(x_1,\dots,x_m,\xi_2,\dots,\xi_n)\otimes W(m,n)_{-1})$. If $n \geq 2$, then the ideal of $S\hat{\otimes}\mathcal{O}(m,n)$ generated by $S\xi_2$ is a proper μ-stable ideal of $S\hat{\otimes}\mathcal{O}(m,n)$, contradicting the minimality of $S\hat{\otimes}\mathcal{O}(m,n)$. It follows that $n=1$. We shall now prove that $m=0$. Indeed, if D_0 lies in $W(m,1)_{\geq 0}$, then $m=0$, otherwise the ideal generated by Sx_1 would be a proper μ-stable ideal of $S\hat{\otimes}\mathcal{O}(m,1)$. Now suppose that $m \geq 1$ and that D_0 has non-zero projection on $W(m,1)_{-1}$, i.e., up to a linear change of indeterminates, $D_0 = \frac{\partial}{\partial x_1} + \delta_0$ for some $\delta_0 \in W(m,0)_{\geq 0}$ and $\mu' = \frac{\partial}{\partial \xi_1} + \xi_1 \frac{\partial}{\partial x_1} + \xi_1\delta_0 + D_1$. Since μ lies in $Lie_1(J)$, we have: $\deg(\xi_1) = -1$ and $\deg(x_1) = -2$, but this is a contradiction since the $\mathbb{Z}$-grading of $Lie(J,\mu)$ has depth 1. We thus have proved that $m=0$, i.e., $Lie(J,\mu) = S\otimes\Lambda(1) + \mathbb{F}\mu + \mathbb{F}[\mu,\mu]$. Notice that either $\mu = \delta\otimes 1 + \alpha\otimes\frac{\partial}{\partial\xi_1}$ for some $\delta \in (DerS)_{\bar{1}}$ and some $\alpha \in \mathbb{F}$ (resp. $\mu = d\otimes\xi_1$ for some $d \in (DerS)_{\bar{0}}$), hence $[\mu,\mu]=0$ and, by Proposition 6.5, (c') holds, or $\mu = d\otimes\xi_1 + 1\otimes\frac{\partial}{\partial\xi_1}$ for some even derivation $d \in DerS$, hence $[\mu,\mu] = 2d$, and $Lie(J,\mu) = S\otimes\Lambda(1) + \mathbb{F}\mu + \mathbb{F}d$. If d is an inner derivation of S we are in case (c'), otherwise (d') holds. □

8 Classification of admissible gradings

The following proposition lists the $\mathbb{Z}$-gradings of depth 1 of all simple linearly compact Lie superalgebras. Following [10] we denote a finite-dimensional contragredient Lie superalgebra by $G(A,\tau)/C$ (where C is a central ideal of $G(A,\tau)$), we denote its standard Chevalley generators by e_i and f_i, and let $\sum_i a_i\alpha_i$ be the highest root. Likewise, we will denote by e_i, f_i the standard Chevalley generators of $q(n)_{\bar{0}}$ and by $\bar{e}_i$, $\bar{f}_i$ the corresponding elements in $q(n)_{\bar{1}}$. Besides, we will identify $p(n)$ with the subalgebra of the Lie superalgebra $SHO(n,n)$ spanned by the following elements: $\{x_ix_j, \xi_i\xi_j : i,j = 1,\dots,n;\ x_i\xi_j : i \neq j = 1,\dots,n;\ x_i\xi_i - x_{i+1}\xi_{i+1} : i = 1,\dots,n-1\}$ (cf. [5, §1.3]), and thus describe the $\mathbb{Z}$-gradings of $p(n)$ as induced by $\mathbb{Z}$-gradings of $SHO(n,n)$.

Proposition 8.1 *All $\mathbb{Z}$-gradings of depth 1 of all simple linearly compact Lie superalgebras S are, up to isomorphism, the following:*

1. $S=G(A,\tau)/C$*:* $\deg e_i=-\deg f_i=k_i$ *with* $k_s=1$ *for* s *such that* $a_s=1$ *and* $k_i=0$ *for every* $i\neq s$*;*

2. $S=q(n)$*:* $\deg(e_i)=\deg(\bar{e}_i)=-\deg(f_i)=-\deg(\bar{f}_i)=k_i$ *with* $k_s=1$ *for some* s *and* $k_i=0$ *for every* $i\neq s$*;*

3. $S=p(n)$*,* $n\geq 2$*:*

(a) $(1,0,\dots,0|-1,0,\dots,0)$*;*

(b) $(\underbrace{1,\dots,1}_{h},0,\dots,0|\underbrace{0,\dots,0}_{h},1,\dots,1)$ *with* $h=0,\dots,n$*;*

4. $S=W(m,n)$*,* $(m,n)\neq(0,1)$*;* $S(m,n)$*,* $m>1$*, or* $m=0$ *and* $n\geq 3$*, or* $m=1$ *and* $n\geq 2$*:*

(a) $(\underbrace{1,\dots,1}_{h},0,\dots,0|\underbrace{1,\dots,1}_{k},0,\dots,0)$ *with* $h=0,\dots,m$*,* $k=0,\dots,n$*;*

(b) $(0,\dots,0|-1,0,\dots,0)$*;*

5. $S=H(2k,n)$*,* $k\geq 1$*, or* $k=0$ *and* $n\geq 4$*:*

(a) $(1,\dots,1|1,\dots,1)$*;*

(b) $n\geq 2$*,* $(0,\dots,0|1,0,\dots,0,-1)$*;*

(c) $n=2t$*,* $(\underbrace{1,\dots,1}_{h},\underbrace{0,\dots,0}_{k-h},\underbrace{0,\dots,0}_{h},1,\dots,1|\underbrace{1,\dots,1}_{t},0,\dots,0)$ *with* $h=0,\dots,k$*;*

6. $S=K(2k+1,n)$*:*

(a) $n\geq 2$*,* $(0,\dots,0|1,0,\dots,0,-1)$*;*

(b) $n=2t$*,* $(1,\underbrace{1,\dots,1}_{h},\underbrace{0,\dots,0}_{k-h},\underbrace{0,\dots,0}_{h},1,\dots,1|\underbrace{1,\dots,1}_{t},0,\dots,0)$ *with* $h=0,\dots,k$*;*

7. $S=HO(n,n)$*,* $n\geq 2$*,* $SHO(n,n)$*,* $n\geq 3$*:*

(a) $(1,\dots,1|1,\dots,1)$*;*

(b) $(0,\dots,0,1|0,\dots,0,-1)$*;*

(c) $(\underbrace{1,\dots,1}_{h},0,\dots,0|\underbrace{0,\dots,0}_{h},1,\dots,1)$ *with* $h=0,\dots,n$*;*

8. $S=KO(n,n+1)$*,* $n\geq 2$*,* $SKO(n,n+1;\beta)$*,* $n\geq 2$*:*

(a) $(0,\dots,0,1|0,\dots,0,-1,0)$*;*

(b) $(\underbrace{1,\dots,1}_{h},0,\dots,0|\underbrace{0,\dots,0}_{h},1,\dots,1,1)$ *with* $h=0,\dots,n$*;*

9. $S=SKO(2,3;0)$*:*

(a) $(1,1|-1,-1,0)$*;*

10. $S=SHO^{\sim}(n,n)$*,* $n\geq 2$*, even:*

(a) $(1,\ldots,1|0,\ldots,0)$;

11. $S=E(5,10)$:

(a) $(1,1,1,1,0)$;

(b) $(1,1,0,0,0)$;

12. $S=E(4,4)$:

(a) $(1,1,1,1)$;

(b) $(1,1,0,0)$;

13. $S=E(3,6)$:

(a) $(1,0,0;\frac{1}{2})$;

(b) $(1,1,0;0)$;

(c) $(1,1,1;\frac{1}{2})$;

14. $S=E(3,8)$:

(a) $(1,0,0;0)$;

(b) $(1,0,0;-1)$;

(c) $(1,1,0;-1)$;

15. $S=E(1,6)$:

(a) $(0|1,0,0,-1,0,0)$;

(b) $(1|1,1,1,0,0,0)$;

(c) $(1|1,1,0,0,0,1)$.

Proof. If S is a finite-dimensional contragredient Lie superalgebra, any $\mathbb{Z}$-grading of S is defined by setting $\deg(e_i)=-\deg(f_i)=k_i\in\mathbb{Z}_{\geq 0}$. Then it is clear that such a grading has depth one if and only if all k_i's are 0 except for $k_s=1$ for some s such that $a_s=1$. Likewise, one gets the statement for $S=q(n)$, using that $q(n)_{\bar{0}}\cong A_n$.

Recall that any $\mathbb{Z}$-grading of finite depth of $W(m,n)$ is isomorphic to that of type $(a_1,\ldots,a_m|b_1,\ldots,b_n)$ with $a_i\in\mathbb{Z}_{\geq 0}$ [5], [3]. Suppose that it has depth 1. Then $\deg\frac{\partial}{\partial x_i}=-a_i\geq -1$ and $\deg\frac{\partial}{\partial\xi_j}=-b_j\geq -1$, i.e., the grading is isomorphic to a grading of type $(1,\ldots,1,0,\ldots,0|b_1,\ldots,b_n)$ with h 1's and $m-h$ 0's, for some $h=0,\ldots,m$ and $b_j\leq 1$. If $n=0$, then the statement for $S=W(m,n)$ is proved. Let $n>0$ and assume $m\geq 1$. If $h\geq 1$, then $b_j\geq 0$, since $\deg\xi_j\frac{\partial}{\partial x_1}=b_j-1$, i.e., the grading is isomorphic to the grading of type $(1,\ldots,1,0,\ldots,0|1,\ldots,1,0,\ldots,0)$ with $h+k$ 1's, for some $h=1,\ldots,m$ and some $k=0,\ldots,n$. If $m\geq 1$ and $h=0$, then $b_j\geq -1$, since $\deg\xi_j\frac{\partial}{\partial x_1}=b_j$. In this case, either $b_j\geq 0$ for every $j=1,\ldots,n$, and we are in the same situation as above, or there exists some s such that $b_s=-1$. Since $\deg\xi_s\frac{\partial}{\partial\xi_t}=b_s-b_t$, it follows that $b_t=0$ for every $t\neq s$, i.e. the grading is isomorphic to the grading of type $(0,\ldots,0|-1,0,\ldots,0)$. Finally, if $m=0$ and $n\geq 2$, then $b_i\geq -1$ and $b_i-b_j\geq -1$, since $\deg\xi_i\xi_j\frac{\partial}{\partial\xi_j}=b_i$ and $\deg\xi_i\frac{\partial}{\partial\xi_j}=b_i-b_j$. It follows that the grading is isomorphic either to the grading of type $(1,\ldots,1,0,\ldots,0)$ with h 1's,

for some $h = 1, \dots, n$, or to the grading of type $(-1, 0, \dots, 0)$. Note that, if $n = 2$, then the grading of type $(-1, -1)$ is isomorphic to the grading of type $(1, 1)$, and if $n \geq 3$, then $b_i + b_j \geq -1$, since $\deg \xi_i \xi_j \xi_k \frac{\partial}{\partial \xi_k} = b_i + b_j$.

The arguments for the closed subalgebras of $W(m, n)$ are similar.

If S is a simple infinite-dimensional exceptional Lie superalgebra, then one uses the description of the $\mathbb{Z}$-gradings of S given in [5] and [3]. For example, let $S = E(5, 10)$. Then $E(5, 10)_{\bar{0}} \cong S_5$ and $E(5, 10)_{\bar{1}} \cong d\Omega^1(5)$ as an S_5-module. A $\mathbb{Z}$-grading of finite depth of $E(5, 10)$ is defined by setting $\deg x_i = -\deg \frac{\partial}{\partial x_i} = a_i \in \mathbb{Z}_{\geq 0}$ and $\deg d = -1/4 \sum_{i=1}^5 a_i \in \frac{1}{2}\mathbb{Z}$. Therefore, if such a grading has depth one, then $a_i \leq 1$, hence the statement holds, since $\sum_{i=1}^5 a_i \in 2\mathbb{Z}$.

Similar arguments apply to $S = E(4, 4), E(3, 6), E(3, 8)$ and $E(1, 6)$. $\square$

Remark 8.2 Let us consider the Lie superalgebra $SKO(2, 3; \beta)$. For every $\beta \neq 0, -1$ we have: $SKO(2, 3; \beta)_{\bar{0}} \cong W(2, 0)$, $SKO(2, 3; \beta)_{\bar{1}} \cong \Omega^0(2)^{-\frac{1}{\beta+1}} \oplus \Omega^0(2)^{-\frac{\beta}{\beta+1}}$ [3, Remark 4.15]. Let us consider $W(2, 0)$ and $\Omega^0(2)$ with the grading of type $(0, 1|)$, so that $W(2, 0) = \prod_{j \geq -1} W(2, 0)_j$ and $\Omega^0(2) = \prod_{j \geq 0} \Omega^0(2)_j$. Denote by $\mathfrak{g} = \prod_{j \geq -1} \mathfrak{g}_j$ and $\mathfrak{g} = \prod_{j \geq -1} \mathfrak{h}_j$ the Lie superalgebra $SKO(2, 3; \beta)$ with the gradings of type $(0, 1|1, 0, 1)$ and $(0, 1|0, -1, 0)$, respectively. Then, for every $\beta \neq 0, -1$ we have: $(\mathfrak{g}_j)_{\bar{0}} = (\mathfrak{h}_j)_{\bar{0}} \cong W(2, 0)_j$, $(\mathfrak{g}_j)_{\bar{1}} \cong \Omega^0(2)_j^{-\beta/(\beta+1)} \oplus \Omega^0(2)_{j+1}^{-1/(\beta+1)}$ and $(\mathfrak{h}_j)_{\bar{1}} \cong \Omega^0(2)_j^{-1/(\beta+1)} \oplus \Omega^0(2)_{j+1}^{-\beta/(\beta+1)}$ $(\Omega^0(2)_{-1} = 0)$. It follows that for every $\beta \neq 0, -1$, $SKO(2, 3; \beta)$ with the grading of type $(0, 1|0, -1, 0)$ is isomorphic to $SKO(2, 3; 1/\beta)$ with the grading of type $(0, 1|1, 0, 1)$.

If $\beta = -1$, then $SKO(2, 3; -1)_{\bar{0}}$ is not simple: it has a commutative ideal isomorphic to $\Omega^0(2)$. We have: $SKO(2, 3; -1)_{\bar{0}} \cong \Omega^0(2) \rtimes S(2, 0)$ and $SKO(2, 3; -1)_{\bar{1}} \cong \Omega^0(2)_+ \oplus \Omega^0(2)_-$, where $\Omega^0(2)_+$ and $\Omega^0(2)_-$ are two odd copies of $\Omega^0(2)$ on which $S(2, 0)$ acts in the natural way. The even functions in $\Omega^0(2)$ act by multiplication on $\Omega^0(2)_+$ and by $-$multiplication on $\Omega^0(2)_-$ [3, Remark 4.19]. Besides, if $f_+ \in \Omega^0(2)_+$ and $g_- \in \Omega^0(2)_-$, then $[f_+, g_-] = \frac{\partial f_+}{\partial x_1} \frac{\partial g_-}{\partial x_2} - \frac{\partial f_+}{\partial x_2} \frac{\partial g_-}{\partial x_1} - (f_+ dg_- + g_- df_+)$. Notice that if f_- and g_+ are the corresponding elements in $\Omega^0(2)_-$ and $\Omega^0(2)_+$, respectively, then $[f_-, g_+] = -\frac{\partial f_-}{\partial x_1} \frac{\partial g_+}{\partial x_2} + \frac{\partial f_-}{\partial x_2} \frac{\partial g_+}{\partial x_1} - (f_- dg_+ + g_+ df_-)$.

As above, let us denote by $\mathfrak{g} = \prod_{j \geq -1} \mathfrak{g}_j$ and $\mathfrak{g} = \prod_{j \geq -1} \mathfrak{h}_j$ the Lie superalgebra $SKO(2, 3; -1)$ with respect to the gradings of type $(0, 1|1, 0, 1)$ and $(0, 1|0, -1, 0)$, respectively. We have: $(\mathfrak{g}_j)_{\bar{0}} = (\mathfrak{h}_j)_{\bar{0}} \cong S(2, 0)_j + \Omega^0(2)_j$, $(\mathfrak{g}_j)_{\bar{1}} \cong (\Omega^0(2)_-)_j \oplus (\Omega^0(2)_+)_{j+1}$ and $(\mathfrak{h}_j)_{\bar{1}} \cong (\Omega^0(2)_+)_j \oplus (\Omega^0(2)_-)_{j+1}$. It follows that the map $\Phi : \mathfrak{g} = \prod \mathfrak{g}_j \longrightarrow \mathfrak{g} = \prod \mathfrak{h}_j$, $\Phi(a) = -a$ for $a \in \Omega^0(2)$, $\Phi(X) = X$ for $X \in S(2, 0)$, $\Phi(f_+) = f_-$ for $f_+ \in (\Omega^0(2)_+)$, $\Phi(f_-) = f_+$ for $f_- \in (\Omega^0(2)_-)$, is an isomorphism of $\mathbb{Z}$-graded Lie superalgebras.

Theorem 8.3 *i) An even admissible grading of a simple linearly compact Lie superalgebra is either short or isomorphic to one of the following:*

1. $S = W(0, 3)$, $(|1, 1, 0)$;
2. $S = W(0, 4)$, $(|1, 1, 0, 0)$;
3. $S = S(0, 4)$, $(|1, 1, 0, 0)$;
4. $S = S(0, 5)$, $(|1, 1, 0, 0, 0)$;
5. $S = S(1, 4)$, $(0|1, 1, 0, 0)$;
6. $S = S(2, 0)$, $(1, 1|)$.

7. $S = S(2,0)$, $(1,0|)$.

ii) An odd admissible grading of a simple linearly compact Lie superalgebra is either short or isomorphic to one of the following:

1. $S = W(1,1)$, $(1|0)$;

2. $S = W(1,2)$, $(0|1,1)$;

3. $S = W(2,1)$, $(1,0|0)$;

4. $S = W(2,2)$, $(0,0|1,1)$;

5. $S = S(1,3)$, $(0|1,1,1)$;

6. $S = S(2,2)$, $(0,0|1,1)$;

7. $S = H(2,2)$, $(1,0|1,0)$;

8. $S = HO(2,2)$, $(1,0|0,1)$;

9. $S = HO(3,3)$, $(0,0,0|1,1,1)$;

10. $S = HO(n+1,n+1)$ *with* $n \geq 1$, $(0,\dots,0,1|0,\dots,0,-1)$;

11. $S = SHO(3,3)$, $(1,0,0|0,1,1)$;

12. $S = SHO(4,4)$, $(0,0,0,0|1,1,1,1)$;

13. $S = SHO(n+1,n+1)$ *with* $n \geq 2$, $(0,\dots,0,1|0,\dots,0,-1)$;

14. $S = KO(n+1,n+2)$ *with* $n \geq 1$, $(0,\dots,0,1|0,\dots,0,-1,0)$;

15. $S = KO(2,3)$, $(0,0|1,1,1)$;

16. $S = SKO(2,3;\beta)$, $(1,0|0,1,1)$;

17. $S = SKO(2,3;\beta)$, $(0,1|0,-1,0)$;

18. $S = SKO(3,4;\beta)$, $(0,0,0|1,1,1,1)$;

19. $S = SKO(n,n+1;\beta)$ *with* $n > 2$ *and* $\beta \neq \frac{4}{n}$, $(0,\dots,0,1|0,\dots,0,-1,0)$.

Proof. By Remark 6.2, among the $\mathbb{Z}$-gradings of depth 1 of all simple linearly compact Lie superalgebras $S = \prod_{j\geq -1} S_j$ listed in Proposition 8.1, we select the non-short ones such that either the dimension of S_0 is infinite and growth $S_1 \leq$ growth S_0 and size $S_1 \leq$ size S_0, or the dimension of S_0 is finite and $\dim S_1 \leq \dim S_0 + 1$. Up to isomorphism, we get a grading in the following list:

A) $S = p(n)$, $(1,0,\dots,0|-1,0,\dots,0)$;

B) $S = W(0,3)$, $(|1,1,1)$;

C) $S = W(0,3)$ or $S = S(0,3)$, $(|1,1,0)$;

D) $S = W(0,4)$ or $S = S(0,4)$, $(|1,1,0,0)$;

E) $S = W(1,2)$, $(0|1,1)$;

F) $S = W(1,3)$ or $S = S(1,3)$, $(0|1,1,0)$;

G) $S = W(2,2)$ or $S = S(2,2)$, $(0,0|1,1)$;

H) $S = W(m,n)$ or $S = S(m,n)$ with $m \geq 1$, $(1,0,\dots,0|0,\dots,0)$;

I) $S = S(0,4)$, $(|1,1,1,0)$;

J) $S = S(0,5)$, $(|1,1,0,0,0)$;

K) $S = S(1,2)$, $(1|1,1)$;

L) $S = S(1,3)$, $(0|1,1,1)$;

M) $S = S(1,4)$, $(0|1,1,0,0)$;

N) $S = S(2,3)$, $(0,0|1,1,0)$;

O) $S = S(3,2)$, $(0,0,0|1,1)$;

P) $S = H(0,5)$, $(|1,1,1,1,1)$;

Q) $S = S(2,0)$, $(1,1|)$;

R) $S = H(2,2)$, $(1,0|1,0)$;

S) $S = H(0,6)$, $(|1,1,1,0,0,0)$;

T) $S = HO(n,n)$ with $n \geq 2$ and $S = SHO(n,n)$ with $n \geq 3$, $(0,\dots,0,1|0,\dots,0,-1)$;

U) $S = HO(2,2)$, $(1,0|0,1)$;

V) $S = HO(3,3)$, $(0,0,0|1,1,1)$;

W) $S = SHO(3,3)$, $(1,0,0|0,1,1)$;

X) $S = SHO(4,4)$, $(0,0,0,0|1,1,1,1)$;

Y) $S = KO(n,n+1)$ and $S = SKO(n,n+1,\beta)$ with $n \geq 2$, $(0,\dots,0,1|0,\dots,0,-1,0)$;

Z) $S = KO(2,3)$, $(0,0|1,1,1)$;

Z′) $S = SKO(3,4;\beta)$, $(0,0,0|1,1,1,1)$.

Note that the grading of type $(1|1)$ and the grading of type $(1|0)$ of $W(1,1)$ are isomorphic. Moreover, in the above list we omitted the Lie superalgebra $H(2,0)$, since it is isomorphic to $S(2,0)$, the Lie superalgebras $KO(1,2)$ and $K(1,2)$, since they are isomorphic to $W(1,1)$, and the Lie superalgebra $S(2,1)$ since it is isomorphic to $HO(2,2)$. Besides, $SKO(2,3;0) \cong HO(2,2)$ and, due to Remark 8.2, for every $\beta \neq 0$ the Lie superalgebra $SKO(2,3;\beta)$ with the grading of type $(0,1|1,0,1)$ is isomorphic to the Lie superalgebra $SKO(2,3;\frac{1}{\beta})$ with the grading of type $(0,1|0,-1,0)$.

All gradings A)–Z′) are transitive and satisfy property (6.3), with the exception of the grading of type $(1|)$ of $W(1,0)$, since in this case S_1 is one-dimensional hence $S_2 \neq S_1^2$. In order to prove that one of these gradings is even admissible (resp. odd admissible) it is therefore sufficient to show that there exists an element $\mu \in (S_1)_{\bar{0}}$ (resp. $\mu \in (S_1)_{\bar{1}}$) satisfying properties (6.1) and (6.2). In Tables 1 and 2 we list, up to isomorphism, all even admissible and odd admissible gradings and indicate such an element μ for each of them. One shows that properties (6.1) and (6.2) hold by

direct computation, as in Example 8.7. For notation concerning finite-dimensional and infinite-dimensional Lie superalgebras, not introduced in this paper, we refer to [10] and [3], respectively.

In all cases not appearing in Tables 1 and 2, either property (6.1) or property (6.2) fails for every even or odd element in S_1. For instance, let $S = W(m,n)$ with $m \geq 1$ and $n \geq 1$, and consider the grading of S of type $(1,0,\dots,0|0,\dots,0)$. Then $S_{-1} = \langle \frac{\partial}{\partial x_1} \rangle \otimes A$, where $A = \mathbb{F}[[x_2,\dots,x_m]] \otimes \Lambda(n)$, $S_0 = \langle x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_j} | i = 2,\dots,m, j = 1,\dots,n \rangle \otimes A$ and $S_1 = \langle x_1^2 \frac{\partial}{\partial x_1}, x_1 \frac{\partial}{\partial x_i}, x_1 \frac{\partial}{\partial \xi_j} | i = 2,\dots,m, j = 1,\dots,n \rangle \otimes A$. It follows that for every $\mu \in (S_1)_{\bar{0}}$, the subalgebra of S_0 generated by $[S_{-1},\mu]$ does not contain the vector fields $\frac{\partial}{\partial \xi_j}$, hence $[S_{-1},\mu]$ does not generate S_0. The same kind of argument applies to the gradings A), C) for $S = S(0,3)$, F) for $S = S(1,3)$, H) for $S = W(m,0)$ with $m \geq 2$, and $S = S(m,n)$ unless $m = 2$, $n = 0$, K), I), N), R), S), T), U), W), hence these gradings are not even admissible. Note that the 1st graded component of S with respect to gradings B), E), G), L), O), P), V), X), Z), Z′) is completely odd.

Let $S = W(1,3)$ with the grading of type $(0|1,1,0)$. Then $S_{-1} = \langle \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} \rangle \otimes A$ with $A = \mathbb{F}[[x]] \otimes \Lambda(\xi_3)$, $S_0 = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial \xi_3}, \xi_i \frac{\partial}{\partial \xi_j} | i,j = 1,2 \rangle \otimes A$, $S_1 = \langle \xi_i \frac{\partial}{\partial x}, \xi_i \frac{\partial}{\partial \xi_3}, \xi_1\xi_2 \frac{\partial}{\partial \xi_i} | i = 1,2 \rangle \otimes A$. S_0 and S_1 are therefore infinite-dimensional subspaces of growth 1 and size 12. One shows that if μ lies in $(S_1)_{\bar{0}}$, then the centralizer of μ in S_0 has size strictly greater than 0, therefore condition (6.1) of Definition 6.1 cannot hold and the grading is not even admissible. We thus get the list of cases in *i*).

Let $S = W(0,3)$ with the grading of type $(|1,1,0)$. Then $S_{-1} = \langle \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} \rangle \otimes A$ with $A = \Lambda(\xi_3)$, $S_0 = \langle \xi_i \frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_3} | i,j = 1,2 \rangle \otimes A$ and $S_1 = \langle \xi_1\xi_2 \frac{\partial}{\partial \xi_i}, \xi_i \frac{\partial}{\partial \xi_3} | i = 1,2 \rangle \otimes A$. It follows that, for every $\mu \in (S_1)_{\bar{1}}$, the subalgebra of S_0 generated by $[S_{-1},\mu]$ does not contain the vector field $\frac{\partial}{\partial \xi_3}$ lying in S_0, therefore the grading is not odd admissible. The same kind of argument applies to the gradings A), C), D), F), H) unless $(m,n) = (1,1)$ or $(m,n) = (2,1)$, I), J), M), N), O), S), hence these gradings are not odd admissible. Notice that the superalgebra $S = S(2,0)$ in Q) is completely even.

Let $S = H(0,5)$ with the principal grading, i.e., the grading of type $(|1,1,1,1,1)$. Then $S_{-1} = \langle \xi_i | i = 1,\dots,5 \rangle$, $S_0 = \langle \xi_i\xi_j | i,j = 1,\dots,5 \rangle$ and $S_1 = \langle \xi_i\xi_j\xi_k | i,j,k = 1,\dots,5 \rangle$. It follows that $\dim(S_0) = \dim(S_1) = 10$. One shows that, for every choice of $\mu \in (S_1)_{\bar{1}}$, μ lies in $[S_0,\mu]$ and the centralizer of μ in S_0 has positive dimension. It follows that condition (6.1) of Definition 6.1 cannot be satisfied, hence the grading is not odd admissible. Similarly, consider $S = W(0,3)$ with the grading of type $(|1,1,1)$. Then $S_0 \cong gl_3$ and $S_1 \cong \mathbb{F}^3 \oplus S^2(\mathbb{F}^3)^*$, where $\mathbb{F}^3$ denotes the standard gl_3-module. The stabilizer of a generic point in $S^2(\mathbb{F}^3)^*$ is isomorphic to so_3 and the stabilizer of so_3 in $\mathbb{F}^3$ is isomorphic to $\mathbb{F}$. It follows that the codimension of $[S_0,\mu]$ in S_1, for a generic point $\mu \in S_1$, is equal to one. Since $\mu \in [S_0,\mu]$, condition (6.1) cannot be satisfied, hence the grading is not odd admissible. The same kind of argument applies to K) and Y) for $S = SKO(n,n+1;4/n)$ when $n \geq 2$, therefore these gradings are not odd admissible. We thus get the list in *ii*). □

In Theorem 8.8 we will prove that if $\mathfrak{g} = \prod_{j \geq -1} \mathfrak{g}_j$ is a simple linearly compact Lie superalgebra with an admissible grading, then Tables 1 and 2 below provide a complete list, up to an automorphism from $exp(ad(\mathfrak{g}_{int}))$, where $\mathfrak{g}_{int}$ is the subalgebra of $(\mathfrak{g}_0)_{\bar{0}}$, generated by its elements which are ad-exponentiable in $\mathfrak{g}$, of even and odd elements $\mu \in \mathfrak{g}_1$, satisfying properties (6.1) and (6.2).

S	type of grading	μ
$W(0,3)$	$(\|1,1,0)$	$\xi_1\frac{\partial}{\partial\xi_3}+\xi_1\xi_2\xi_3(\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\xi_2})$
$W(0,4)$	$(\|1,1,0,0)$	$P_W=\xi_1(\frac{\partial}{\partial\xi_3}+\xi_2\xi_3\frac{\partial}{\partial\xi_1}+\xi_2\xi_3\frac{\partial}{\partial\xi_2})+\xi_2(\frac{\partial}{\partial\xi_4}+\xi_3\xi_4\frac{\partial}{\partial\xi_3})$
$S(0,4)$	$(\|1,1,0,0)$	$P_W+\xi_1\xi_3\xi_4\frac{\partial}{\partial\xi_4}+\xi_1\xi_2\xi_4\frac{\partial}{\partial\xi_1}-\xi_2\xi_3\xi_4\frac{\partial}{\partial\xi_4}$
$S(0,5)$	$(\|1,1,0,0,0)$	$\xi_1(\frac{\partial}{\partial\xi_3}+\xi_3\xi_4\frac{\partial}{\partial\xi_5})+\xi_2(\frac{\partial}{\partial\xi_4}+\xi_3\xi_4\frac{\partial}{\partial\xi_3}+\xi_4\xi_5\frac{\partial}{\partial\xi_5}+\xi_3\xi_4\frac{\partial}{\partial\xi_5})$
$S(1,4)$	$(0\|1,1,0,0)$	$\xi_1\frac{\partial}{\partial\xi_3}+\xi_2\frac{\partial}{\partial\xi_4}+\xi_1\xi_3\frac{\partial}{\partial x}+\alpha\xi_1\xi_4\frac{\partial}{\partial x}+\xi_2\xi_4(x\frac{\partial}{\partial x}+\xi_3\frac{\partial}{\partial\xi_3}),\ \alpha\in\mathbb{F}$
$S(2,0)$	$(1,1\|)$	$x_1^2\frac{\partial}{\partial x_2}+x_2^2\frac{\partial}{\partial x_1}$
$S(2,0)$	$(1,0\|)$	$x_1\frac{\partial}{\partial x_2}$

Table 1: even admissible gradings

S	type of grading	$\mu\ \ (\alpha\in\mathbb{F})$
$W(1,1)$	$(1\|0)$	$x^2\xi\frac{\partial}{\partial x}+x\frac{\partial}{\partial\xi}$
$W(1,2)$	$(0\|1,1)$	$\xi_1\frac{\partial}{\partial x}+(\alpha+x)\xi_1\xi_2\frac{\partial}{\partial\xi_1}$
$W(2,1)$	$(1,0\|0)$	$x_1^2\xi\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial\xi}+x_1\xi\frac{\partial}{\partial x_2}$
$W(2,2)$	$(0,0\|1,1)$	$\xi_1\frac{\partial}{\partial x_1}+\xi_2\frac{\partial}{\partial x_2}+\xi_1\xi_2((1+x_1)\frac{\partial}{\partial\xi_1}+\alpha x_2\frac{\partial}{\partial\xi_2})$
$S(2,2)$	$(0,0\|1,1)$	$\xi_1\frac{\partial}{\partial x_1}+(1+\alpha x_1+x_1x_2)\xi_2\frac{\partial}{\partial x_2}-x_1\xi_1\xi_2\frac{\partial}{\partial\xi_1}$
$S(1,3)$	$(0\|1,1,1)$	$\xi_3\frac{\partial}{\partial x}+\xi_1\xi_2\frac{\partial}{\partial\xi_3}$
$H(2,2)$	$(1,0\|1,0)$	$(\alpha+q)p^2\xi_2+p\xi_1$
$HO(2,2)$	$(1,0\|0,1)$	$x_1^2+x_1\xi_1\xi_2$
$HO(3,3)$	$(0,0,0\|1,1,1)$	$\xi_1\xi_2+(\alpha+x_1)x_1\xi_1\xi_3$
$HO(n,n)$ $_{(n\geq 2)}$	$(0,\dots,0,1\|0,\dots,0,-1)$	$x_n+x_n^2\xi_n\xi_{n-1}$
$SHO(3,3)$	$(1,0,0\|0,1,1)$	$x_1^2+\xi_2\xi_3+x_1\xi_3\xi_1-x_2\xi_3\xi_2$
$SHO(4,4)$	$(0,0,0,0\|1,1,1,1)$	$\xi_1\xi_2+(\alpha+x_1)\xi_3\xi_4$
$SHO(n,n)$ $_{(n\geq 3)}$	$(0,\dots,0,1\|0,\dots,0,-1)$	$x_n+x_n^2\xi_n\xi_{n-1}+2x_nx_{n-2}\xi_{n-1}\xi_{n-2}$
$KO(n,n+1)$ $_{(n\geq 2)}$	$(0,\dots,0,1\|0,\dots,0,-1,0)$	$x_n+x_n^2\xi_n\xi_{n-1}$
$KO(2,3)$	$(0,0\|1,1,1)$	$\xi_1\tau+x_1\xi_1\xi_2$
$SKO(n,n+1;\beta)$ $_{(n\geq 2,\ \ \beta\neq 4/n)}$	$(0,\dots,0,1\|0,\dots,0,-1,0)$	$x_n+x_n\xi_{n-1}(\tau-\frac{2-n\beta}{2}x_n\xi_n)$
$SKO(2,3;\beta)$ $_{(\beta\neq 0,1,2+\frac{1}{b},2+\frac{2}{b},\forall b\in\mathbb{Z}_{>0})}$	$(0,1\|0,-1,0)$	$x_2(1+x_2\xi_2\tau-x_1\xi_1(2\tau-(3-2\beta)x_2\xi_2))$
$SKO(3,4;1)$	$(0,0,1\|0,0,-1,0)$	$x_3(1+\xi_1\xi_2+\xi_2\tau+\frac{1}{2}x_3\xi_2\xi_3)$
$SKO(3,4;\beta)$	$(0,0,0\|1,1,1,1)$	$\xi_1\xi_2+(\alpha+2x_1)(x_1\xi_1\xi_3-\tau\xi_3)-3\beta(\alpha+x_1)x_1\xi_1\xi_3$

Table 2: odd admissible gradings

Lemma 8.4 *Let $S=S_0\supset S_1\supset\dots$ be a filtered linearly compact Lie algebra acting on a linearly compact filtered vector space $V=V_{-k}\supset V_{-k+1}\supset\dots\supset V_0\supset V_1\supset\dots$, such that S_j and V_j have finite codimension and $S_i(V_j)\subset V_{i+j}$. Suppose that*

(8.1) *there exists $\mu\in V$, such that $S(\mu)+\mathbb{F}\mu=V$.*

Then for any other such element μ' there exists an element $a\in\exp(S)\times\mathbb{F}^\times$, such that $a(\mu)=\mu'$.

Proof. Condition (8.1) is equivalent to the existence, for every j, of an element μ_j in V/V_j such that $S(\mu_j)+\mathbb{F}\mu_j=V/V_j$, or, equivalently, such that

$$\mathfrak{s}_j(\mu_j)+\mathbb{F}\mu_j=V/V_j, \tag{8.2}$$

where $\mathfrak{s}_j=S/S_{j+k}$. If μ' is another element satisfying condition (8.1) then there exist elements $\mu'_j\in V/V_j$ such that $\mathfrak{s}_j(\mu'_j)+\mathbb{F}\mu'_j=V/V_j$. From (8.2) it follows that that the orbit of $\mathbb{F}\mu_j$ is Zariski open in V/V_j. Likewise, the orbit of $\mathbb{F}\mu'_j$ is Zariski open in V/V_j. Since any two Zariski open subsets in V/V_j have a non-empty intersection, we conclude that for every j there exists an element $a_j\in\exp(\mathfrak{s}_j)\times\mathbb{F}^\times$ such that $a_j(\mu_j)=\mu'_j$ (see also the proof of [11, Proposition 2]). Then the inverse limit a of the a_j's sends μ to μ'. □

Recall that the *canonical filtration* of a simple infinite-dimensional linearly compact Lie superalgebra $\mathfrak{g}$ is the Weisfeiler filtration of $\mathfrak{g}$, associated to the canonical subalgebra L_0 of $\mathfrak{g}$, i.e., the open maximal subalgebra of $\mathfrak{g}$ equal to the intersection of all open subalgebras of minimal codimension in $\mathfrak{g}$ [3, §11]. The subalgebra L_0 contains all *ad*-exponentiable elements of $\mathfrak{g}$.

Proposition 8.5 *Let $\mathfrak{g}=\prod_{j\geq -1}\mathfrak{g}_j$ be an even (resp. odd) admissible grading of a simple linearly compact Lie superalgebra $\mathfrak{g}$ and let $\mu\in(\mathfrak{g}_1)_{\bar 0}$ (resp. $\mu\in(\mathfrak{g}_1)_{\bar 1}$) be as in Definition 6.1. If*

$$[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu=(\mathfrak{g}_1)_{\bar 0}\quad (\textit{resp. } [\mathfrak{g}_{int},\mu]+\mathbb{F}\mu=(\mathfrak{g}_1)_{\bar 1}), \tag{8.3}$$

then the superalgebra $J(\mathfrak{g},\mu)$ does not depend on the choice of the element μ.

Proof. It is sufficient to use Lemma 8.4 where $S=\mathfrak{g}_{int}$, $V=(\mathfrak{g}_1)_{\bar 0}$ if μ is even or $V=(\mathfrak{g}_1)_{\bar 1}$ if μ is odd, and the filtrations of S and V are induced by the canonical filtration of $\mathfrak{g}$. Since the canonical subalgebra L_0 of $\mathfrak{g}$ contains all its exponentiable elements, $\mathfrak{g}_{int}$ is contained in L_0, hence the filtration of L_0 induces a filtration of $\mathfrak{g}_{int}$ satisfying the hypotheses of Lemma 8.4. □

Lemma 8.6 *Let $\mathfrak{g}=\prod_{j\geq -1}\mathfrak{g}_j$ be a depth 1 $\mathbb{Z}$-graded linearly compact Lie superalgebra, and let $\mathfrak{g}=\prod_{j\geq -k}S_j$ be a $\mathbb{Z}$-grading of $\mathfrak{g}$, such that $\dim(S_j)<\infty$ for all j, and $\mathfrak{g}_{int}\subset\prod_{j\geq 0}S_j$. Let $\mu=\mu_0+h$, where $\mu_0,h\in\mathfrak{g}_1$, $\mu_0\in S_{-k}+\cdots+S_t$, $h\in\prod_{j>t}S_j$, and $[\mathfrak{g}_{int},\mu_0]+(\mathfrak{g}_1\cap\oplus_{j\leq t}S_j)=\mathfrak{g}_1$ for some $t\in\mathbb{Z}_+$. Then $[\mathfrak{g}_{int},\mu]=[\mathfrak{g}_{int},\mu_0]$. Consequently, by Lemma 8.4, μ and μ_0 are conjugate by an element of $exp(ad(\mathfrak{g}_{int}))$.*

Proof. We have: $[\mathfrak{g}_{int},\mu_0]=\sum_{i\in\mathbb{Z}_+}[\mathfrak{g}_{int}\cap S_i,\mu_0]$ and $[\mathfrak{g}_{int},\mu]=\sum_{i\in\mathbb{Z}_+}([\mathfrak{g}_{int}\cap S_i,\mu_0]+[\mathfrak{g}_{int}\cap S_i,h])$, where $[\mathfrak{g}_{int}\cap S_i,h]$ lies in $\oplus_{j>i+t}S_j$. It follows that $[\mathfrak{g}_{int},\mu]=[\mathfrak{g}_{int},\mu_0]$. □

Example 8.7 In Example 2.2 we introduced the Buttin bracket $\{\cdot,\cdot\}$ on $\mathcal{O}(n,n)$. Recall that it induces on the superspace $HO(n,n)=\mathcal{O}(n,n)/\mathbb{F}1$ with reversed parity a simple Lie superalgebra bracket, denoted by $[\cdot,\cdot]$. Consider the Lie superalgebra $\mathfrak{g}=HO(n+1,n+1)$ for $n\geq 1$, with the grading $\mathfrak{g}=\prod_{j\geq -1}\mathfrak{g}_j$ of type $(0,\dots,0,1|0,\dots,0,-1)$ (see §1). Then we have: $\mathfrak{g}_{-1}=\langle\xi_{n+1}\rangle\otimes\mathcal{O}(n,n)$, $\mathfrak{g}_0=(\langle 1,x_{n+1}\xi_{n+1}\rangle\otimes\mathcal{O}(n,n))/\mathbb{F}1$, $\mathfrak{g}_1=\langle x_{n+1},x_{n+1}^2\xi_{n+1}\rangle\otimes\mathcal{O}(n,n)$, with reversed parity, hence we can identify $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ with the following subspaces:

$\mathfrak{g}_{-1}\equiv\mathcal{O}(n,n)$,

$\mathfrak{g}_0\equiv\mathcal{O}(n,n)/\mathbb{F}1\oplus\eta\mathcal{O}(n,n)$ with reversed parity,

$\mathfrak{g}_1 \equiv \mathcal{O}(n,n) \oplus \xi\mathcal{O}(n,n)$ with reversed parity,

where ξ and η are odd indeterminates. Under this identification, the bracket of $\mathfrak{g}_0$ and $\mathfrak{g}_1$ can be written as follows ($f \in \mathcal{O}(n,n)/\mathbb{F}1$, $g, s_1, s_2 \in \mathcal{O}(n,n)$):

$$[f + \eta g, s_1 + \xi s_2] = [f, s_1] + (-1)^{p(g)+1} g s_1 + \xi((-1)^{p(f)+1}[f, s_2] + [g, s_1] - g s_2), \tag{8.4}$$

and the bracket of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ as follows: if ($t \in \mathfrak{g}_{-1}$ has zero constant term, then

$$[t, s_1 + \xi s_2] = (-1)^{p(t)+1} t s_1 + \eta([t, s_1] - 2 t s_2); \tag{8.5}$$

and

$$[1, s_1 + \xi s_2] = -2\eta s_2. \tag{8.6}$$

In order to show that the grading of type $(0, \dots, 0, 1|0, \dots, 0, -1)$ is odd admissible, consider the element $\mu_0 = 1 + \xi\xi_n \in (\mathfrak{g}_1)_{\bar{1}}$. Then (6.1) follows from (8.4). Besides, (8.5) and (8.6) show that the Lie subalgebra generated by $[\mu_0, \mathfrak{g}_{-1}]$ contains the element $\eta\xi_n$ and all elements $t + 2\eta\xi_n t$ with $t \in \mathcal{O}(n,n)$, $t \neq 1$, hence it contains the subalgebra $\eta\mathcal{O}(n,n)$, since $[\eta\xi_n, t + 2\eta\xi_n t] = -\eta\frac{\partial t}{\partial x_n}$, hence it contains $\mathcal{O}(n,n)$. Therefore μ_0 and $\mathfrak{g}_{-1}$ generate $\mathfrak{g}_0$, hence the grading of type $(0, \dots, 0, 1|0, \dots, 0, -1)$ is odd admissible.

We have $J(\mathfrak{g}, \mu_0) = \mathcal{O}(n,n)$ with product ($f, g \in \mathcal{O}(n,n)$):

$$f \bullet g = (-1)^{p(f)+1}\{f, g\} + 2\xi_n f g. \tag{8.7}$$

By Remark 5.3, $J(\mathfrak{g}, \mu_0) \cong OJP(n-1, n-1)$.

We now want to show that the choice of $\mu \in (\mathfrak{g}_1)_{\bar{1}}$ satisfying (6.1) and (6.2) is unique up to automorphisms in $\exp(ad(\mathfrak{g}_{int}))$. Let $\mu = \mu_1 + \xi\mu_2$, with $\mu_1, \mu_2 \in \mathcal{O}(n,n)$, be an odd element in $\mathfrak{g}_1$ satisfying properties (6.1) and (6.2). We shall first show that if μ satisfies (6.2), then μ_1 has a non-zero constant term. In order to construct the Lie subalgebra of $\mathfrak{g}_0$ generated by $[\mathfrak{g}_{-1}, \mu]$, notice that, if $r \in \mathcal{O}(n,n)$, then, by (8.5), $[r, \mu] = (-1)^{p(r)+1} r\mu_1 + \eta t'$ for some $t' \in \mathcal{O}(n,n)$. Besides, if $r', r'', t', t'' \in \mathcal{O}(n,n)$, then:

$$[r'\mu_1 + \eta t', r''\mu_1 + \eta t''] = [r'\mu_1, r''\mu_1] + \eta\omega \tag{8.8}$$

for some $\omega \in \mathcal{O}(n,n)$. Let us denote by I the principal ideal of $\mathcal{O}(n,n)$, with respect to the usual associative product, generated by μ_1. Since $[\mathfrak{g}_{-1}, \mu]$ generate the whole Lie superalgebra $\mathfrak{g}_0$, (8.8) implies that the Lie superalgebra $\mathcal{O}(n,n)/\mathbb{F}1$ is generated by the image $\bar{I}$ of I. Let us now denote by K the principal ideal of $\mathcal{O}(n,n)$, with respect to the usual associative product, generated by $[\mu_1, \mu_1]$, and denote by $\bar{K}$ its image in $\mathcal{O}(n,n)/\mathbb{F}1$. We have: $[I, I] \subset I + K$ and $[I, K] \subset I + K$ since $[\mu_1, [\mu_1, \mu_1]] = 0$ (μ_1 is odd). Finally, $[K, K] \subset K$, hence $I + K$ is a subalgebra of the Lie superalgebra $\mathcal{O}(n,n)$, containing the Lie subalgebra, generated by I, hence $\bar{I} + \bar{K} = \mathcal{O}(n,n)/\mathbb{F}1$. Now suppose that μ_1 has zero constant term, hence, up to a linear change of indeterminates, $\mu_1 = a x_1 + q_1$ where $a \in \mathbb{F}$ and q_1 is a sum of monomials of degree greater than or equal to 2. Then $[\mu_1, \mu_1] = 2a\frac{\partial q_1}{\partial \xi_1} + [q_1, q_1]$, hence $I + K = (x_1 + q_1)\mathcal{O}(n,n) + (2a\frac{\partial q_1}{\partial \xi_1} + [q_1, q_1])\mathcal{O}(n,n)$, therefore it does not contain any even indeterminate different from x_1, since q_1 is an even element of $\mathcal{O}(n,n)$. This contradicts the equality $\bar{I} + \bar{K} = \mathcal{O}(n,n)/\mathbb{F}1$, therefore μ_1 is an invertible element in $\mathcal{O}(n,n)$.

Up to an automorphism, we may thus assume $\mu = 1+\xi\mu_2$. Indeed, if $\mu_1 = 1+\mu_1'$, where μ' has zero constant term,then the automorphism $\exp(ad\eta\mu_1')$ maps μ to an element of the form $1+\xi q_2$, for some q_2 in $\mathcal{O}(n,n)$, which still satisfies both conditions (6.1) and (6.2). By (8.4) and condition (6.2), we thus have:

$$\langle \xi[f,\mu_2],\ g-(-1)^{p(g)+1}\xi g\mu_2 \mid f\in\mathcal{O}(n,n)/\mathbb{F}1, g\in\mathcal{O}(n,n)\rangle + \mathbb{F}\mu = \mathfrak{g}_1, \tag{8.9}$$

hence $\langle [f,\mu_2] \mid f\in\mathcal{O}(n,n)/\mathbb{F}1\rangle = \mathcal{O}(n,n)$. It follows that μ_2 has a non-zero linear term, hence, up to a linear change of indeterminates, since μ_2 is an odd element of $\mathcal{O}(n,n)$, we may assume $\mu_2 = \xi_n+\rho$, where ρ is a sum of monomials of degree greater than or equal to 2, i.e. $\mu = 1+\xi\xi_n+\xi\rho$. In other words, if $\mathfrak{g} = \prod_{j\geq -1} S_j$ denotes the principal grading of $\mathfrak{g}$, then $\mu_0\in S_{-1}+S_2$ and $\mu = \mu_0+h$ with $h\in\prod_{j>2} S_j$. By Lemma 8.6, μ is conjugate to μ_0.

In what follows, if $\mathfrak{g}$ is one of the simple infinite-dimensional linearly compact Lie superalgebras appearing in *i)* and *ii)* of Theorem 8.3, we will denote by $\mathfrak{g} = \prod_j S_j$ the principal grading, defined in [12].

Theorem 8.8 *Let $\mathfrak{g} = \prod_{j\geq -1}\mathfrak{g}_j$ be a simple, linearly compact, infinite-dimensional Lie superalgebra with an admissible grading of height strictly greater than one. Then, up to conjugation by $exp(ad(\mathfrak{g}_{int}))$, a complete list of elements $\mu\in\mathfrak{g}_1$ satisfying properties (6.1) and (6.2), is as follows:*

1. $\mathfrak{g} = W(1,1)$ *with the grading of type* $(1|0)$*:* $\mu = x^2\xi\frac{\partial}{\partial x} + x\frac{\partial}{\partial\xi}$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$.

2. $\mathfrak{g} = W(1,2)$ *with the grading* $(0|1,1)$*:* $\mu = \xi_1\frac{\partial}{\partial x} + (\alpha+x)\xi_1\xi_2\frac{\partial}{\partial\xi_1}$*, where either* $\alpha\neq 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$*, or* $\alpha = 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu$ *has codimension one in* $\mathfrak{g}_1$.

3. $\mathfrak{g} = W(2,1)$ *with the grading* $(1,0|0)$*:* $\mu = x_1\frac{\partial}{\partial\xi} + x_1\xi\frac{\partial}{\partial x_2} + x_1^2\xi\frac{\partial}{\partial x_1}$ *and* $[\mathfrak{g}_{int},\mu] = \mathfrak{g}_1$.

4. $\mathfrak{g} = W(2,2)$ *with the grading* $(0,0|1,1)$*:* $\mu = \xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2} + (1+x_1)\xi_1\xi_2\frac{\partial}{\partial\xi_1} + \alpha x_2\xi_1\xi_2\frac{\partial}{\partial\xi_2}$*, where either* $\alpha\neq 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$*, or* $\alpha = 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu$ *has codimension one in* $\mathfrak{g}_1$.

5. $\mathfrak{g} = S(1,4)$ *with the grading* $(0|1,1,0,0)$*:* $\mu = \xi_1\frac{\partial}{\partial\xi_3} + \xi_2\frac{\partial}{\partial\xi_4} + \alpha\xi_1\xi_4\frac{\partial}{\partial x} + \xi_1\xi_3\frac{\partial}{\partial x} + x\xi_2\xi_4\frac{\partial}{\partial x} - \xi_2\xi_3\xi_4\frac{\partial}{\partial\xi_3}$*, where either* $\alpha\neq 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$*, or* $\alpha = 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu$ *has codimension one in* $\mathfrak{g}_1$.

6. $\mathfrak{g} = S(2,0)$ *with the grading* $(1,1|)$*:* $\mu = x_1^2\frac{\partial}{\partial x_2} + x_2^2\frac{\partial}{\partial x_1}$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$.

7. $\mathfrak{g} = S(2,0)$ *with the grading* $(1,0|)$*:* $\mu = x_1\frac{\partial}{\partial x_2}$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$.

8. $\mathfrak{g} = S(2,2)$ *with the grading* $(0,0|1,1)$*:* $\mu = \xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2} + (\alpha x_1 + x_1x_2)\xi_2\frac{\partial}{\partial x_2} - x_1\xi_1\xi_2\frac{\partial}{\partial\xi_1}$*, where either* $\alpha\neq 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$*, or* $\alpha = 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu$ *has codimension two in* $\mathfrak{g}_1$.

9. $\mathfrak{g} = S(1,3)$ *with the grading* $(0|1,1,1)$*:* $\mu = \xi_3\frac{\partial}{\partial x} + \xi_1\xi_2\frac{\partial}{\partial\xi_3}$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$.

10. $\mathfrak{g} = H(2,2)$ *with the grading* $(1,0|1,0)$*:* $\mu = p\xi_1 + (\alpha+q)p^2\xi_2$*, where either* $\alpha\neq 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$*, or* $\alpha = 0$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu$ *has codimension one in* $\mathfrak{g}_1$.

11. $\mathfrak{g} = HO(2,2)$ *with the grading* $(1,0|0,1)$*:* $\mu = x_1^2 + x_1\xi_1\xi_2$ *and* $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu = \mathfrak{g}_1$.

12. $\mathfrak{g} = HO(3,3)$ *with the grading* $(0,0,0|1,1,1)$*:* $\mu = \xi_1\xi_2 + (\alpha + x_1)x_1\xi_1\xi_3$*, where* $\alpha \neq 0$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu = \mathfrak{g}_1$*, or* $\alpha = 0$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu$ *has codimension one in* $\mathfrak{g}_1$*.*

13. $\mathfrak{g} = HO(n,n)$*,* $n \geq 2$*, with the grading* $(0,\dots,0,1|0,\dots,0,-1)$*:* $\mu = x_n + x_n^2\xi_n\xi_{n-1}$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu = \mathfrak{g}_1$*.*

14. $\mathfrak{g} = SHO(3,3)$ *with the grading* $(1,0,0|0,1,1)$*:* $\mu = x_1^2 + \xi_2\xi_3 + x_1\xi_3\xi_1 - x_2\xi_3\xi_2$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu = \mathfrak{g}_1$*.*

15. $\mathfrak{g} = SHO(4,4)$ *with the grading* $(0,0,0,0|1,1,1,1)$*:* $\mu = \xi_1\xi_2 + (\alpha + x_1)\xi_3\xi_4$*, where either* $\alpha \neq 0$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu = \mathfrak{g}_1$*, or* $\alpha = 0$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu$ *has codimension 1 in* $\mathfrak{g}_1$*.*

16. $\mathfrak{g} = SHO(n,n)$*,* $n \geq 3$*, with the grading* $(0,\dots,0,1|0,\dots,0,-1)$*:* $\mu = x_n + x_n^2\xi_n\xi_{n-1} + 2x_nx_{n-2}\xi_{n-1}\xi_{n-2}$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu = \mathfrak{g}_1$*.*

17. $\mathfrak{g} = KO(n,n+1)$*,* $n \geq 2$*, with the grading* $(0,\dots,0,1|0,\dots,0,-1,0)$*:* $\mu = x_n + x_n^2\xi_n\xi_{n-1}$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu = \mathfrak{g}_1$*.*

18. $\mathfrak{g} = KO(2,3)$ *with the grading* $(0,0|1,1,1)$*:* $\mu = \xi_1\tau + x_1\xi_1\xi_2$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu$ *has codimension one in* $\mathfrak{g}_1$*.*

19. $\mathfrak{g} = SKO(n,n+1;\beta)$*,* $n \geq 2$*,* $\beta \neq \frac{4}{n}$*, with the grading* $(0,\dots,0,1|0,\dots,0,-1,0)$*:* $\mu = x_n + x_n\xi_{n-1}\tau + \frac{2-n\beta}{2}x_n^2\xi_n\xi_{n-1}$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu = \mathfrak{g}_1$ *if* $(n,\beta) \neq (3,1)$*,* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu$ *has codimension one in* $\mathfrak{g}_1$ *if* $n = 3$ *and* $\beta = 1$*.*

20. $\mathfrak{g} = SKO(2,3;\beta)$*,* $\beta \neq 0, 1, 2 + \frac{1}{b}, 2 + \frac{2}{b}$ $\forall b \in \mathbb{Z}_{>0}$*, with the grading* $(0,1|0,-1,0)$*:* $\mu = x_2 + x_2^2\xi_2\tau - 2x_1x_2\xi_1\tau + (3 - 2\beta)x_1x_2^2\xi_1\xi_2$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu$ *has codimension one in* $\mathfrak{g}_1$*.*

21. $\mathfrak{g} = SKO(3,4;1)$*, with the grading* $(0,0,1|0,0,-1,0)$*:* $\mu = x_3 + x_3\xi_1\xi_2 + x_3\xi_2\tau + \frac{1}{2}x_3^2\xi_2\xi_3$ *and* $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu = \mathfrak{g}_1$*.*

22. $\mathfrak{g} = SKO(3,4;\beta)$*, with the grading* $(0,0,0|1,1,1,1)$*:* $\mu = \xi_1\xi_2 + \alpha((1-3\beta)x_1\xi_1\xi_3 - \tau\xi_3) + (2 - 3\beta)x_1^2\xi_1\xi_3 - 2x_1\tau\xi_3$*, where either* $\alpha \neq 0$ *and* $\beta \neq 1/3$ *and* $[\mathfrak{g}_{int}, \mu] = \mathfrak{g}_1$*, or* $\alpha = 0$ *and* $[\mathfrak{g}_{int}, \mu]$ *has codimension one in* $\mathfrak{g}_1$*.*

Proof. If $\mathfrak{g} = W(1,1)$ with the grading of type $(1|0)$, then one checks by direct computation that $\mu = x\frac{\partial}{\partial \xi} + x^2\xi\frac{\partial}{\partial x}$ satisfies properties (6.1) and (6.2). In this case $(\mathfrak{g}_0)_{\bar{0}} = \mathfrak{g}_{int}$, hence 1. follows from Proposition 8.5. The same argument holds in case 6. In all other cases we use the same kind of argument as in Example 8.7, where we prove 12. For instance, let $\mathfrak{g} = S(2,0)$ with the grading of type $(1,0|)$. Then $\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial x_1} \rangle \otimes \mathbb{F}[[x_2]]$, $\mathfrak{g}_0 = \langle x_2^r\frac{\partial}{\partial x_2} - rx_1x_2^{r-1}\frac{\partial}{\partial x_1} \mid r \geq 0 \rangle$ and $\mathfrak{g}_1 = \langle 2x_1x_2^r\frac{\partial}{\partial x_2} - rx_1^2x_2^{r-1}\frac{\partial}{\partial x_1} \mid r \geq 0 \rangle$. Let $\mu = \sum_{r\geq 0}\alpha_r(2x_1x_2^r\frac{\partial}{\partial x_2} - rx_1^2x_2^{r-1}\frac{\partial}{\partial x_1}) \in \mathfrak{g}_1$ satisfy properties (6.1) and (6.2). If $\alpha_0 = 0$, then the subalgebra of $\mathfrak{g}_0$ generated by $[\mathfrak{g}_{-1}, \mu]$ is contained in S_0 hence it is properly contained in $\mathfrak{g}_0$, contradicting property (6.2). It follows that $\alpha_0 \neq 0$, hence, by Lemma 8.6, μ is conjugate to $x_1\frac{\partial}{\partial x_2}$, hence $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu = \mathfrak{g}_1$.

Let us now consider $\mathfrak{g} = W(1,2) = \prod_{j\geq -1}\mathfrak{g}_j$ with the grading of type $(0|1,1)$. We have: $\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial \xi_i} \mid i = 1,2 \rangle \otimes \mathbb{F}[[x]]$, $\mathfrak{g}_0 = \langle \frac{\partial}{\partial x}, \xi_j\frac{\partial}{\partial \xi_i} \mid j, i = 1,2 \rangle \otimes \mathbb{F}[[x]]$, $\mathfrak{g}_1 = \langle \xi_i\frac{\partial}{\partial x}, \xi_1\xi_2\frac{\partial}{\partial \xi_i} \mid i = 1,2 \rangle \otimes \mathbb{F}[[x]]$. Let $\mu \in \mathfrak{g}_1$ satisfy properties (6.1) and (6.2). If $\mu \in \mathfrak{g}_1 \cap \prod_{j\geq 1} S_j$, then the subalgebra of $\mathfrak{g}_0$ generated by $[\mathfrak{g}_{-1}, \mu]$ is contained in $\prod_{j\geq 0} S_j$, hence it is properly contained in $\mathfrak{g}_0$. It follows that, up to a linear change of indeterminates, we may assume $\mu = \xi_1\frac{\partial}{\partial x} + h$ for some $h \in \mathfrak{g}_1 \cap \prod_{j\geq 1} S_j$. We

will show that, up to conjugation by elements in $\exp(ad(\mathfrak{g}_{int}))$, we may assume $h=\alpha\xi_1\xi_2\frac{\partial}{\partial\xi_1}+\varphi$, for some $\alpha\in\mathbb{F}$ and some $\varphi\in\prod_{j\geq 2}S_j$. Indeed, let $\mu=\xi_1\frac{\partial}{\partial x}+a\xi_1\xi_2\frac{\partial}{\partial\xi_1}+b\xi_1\xi_2\frac{\partial}{\partial\xi_2}+cx\xi_1\frac{\partial}{\partial x}+dx\xi_2\frac{\partial}{\partial x}+h_1$ for some $a,b,c,d\in\mathbb{F}$ and some $h_1\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. Then $\exp(ad(-dx\xi_2\frac{\partial}{\partial\xi_1}))(\mu)=\xi_1\frac{\partial}{\partial x}+(a+d)\xi_1\xi_2\frac{\partial}{\partial\xi_1}+b\xi_1\xi_2\frac{\partial}{\partial\xi_2}+cx\xi_1\frac{\partial}{\partial x}+h_1'$ for some $h_1'\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. It follows that we may assume $\mu=\xi_1\frac{\partial}{\partial x}+a\xi_1\xi_2\frac{\partial}{\partial\xi_1}+b\xi_1\xi_2\frac{\partial}{\partial\xi_2}+cx\xi_1\frac{\partial}{\partial x}+h_1$ for some $h_1\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. Then $\exp(ad(bx\xi_2\frac{\partial}{\partial\xi_2}))(\mu)=\xi_1\frac{\partial}{\partial x}+a\xi_1\xi_2\frac{\partial}{\partial\xi_1}+cx\xi_1\frac{\partial}{\partial x}+h_1''$ for some $h_1''\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. It follows that we may assume $\mu=\xi_1\frac{\partial}{\partial x}+a\xi_1\xi_2\frac{\partial}{\partial\xi_1}+cx\xi_1\frac{\partial}{\partial x}+h_1$ for some $h_1\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. Then $\exp(ad(\frac{c}{2}x^2\frac{\partial}{\partial x}))(\mu)=\xi_1\frac{\partial}{\partial x}+a\xi_1\xi_2\frac{\partial}{\partial\xi_1}+\tilde{h}$ for some $\tilde{h}\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. It follows that we may assume $\mu=\xi_1\frac{\partial}{\partial x}+a\xi_1\xi_2\frac{\partial}{\partial\xi_1}+\tilde{h}$ for some $a\in\mathbb{F}$ and $\tilde{h}\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. If $a\neq 0$, then we may assume, up to a linear change of indeterminates, $a=1$. By direct computation one shows that $\mu_0=\xi_1\frac{\partial}{\partial x}+\xi_1\xi_2\frac{\partial}{\partial\xi_1}$ satisfies conditions (6.1) and (6.2), and that $[\mathfrak{g}_0,\mu_0]=[\mathfrak{g}_{int},\mu_0]$, since $[\frac{\partial}{\partial x},\mu_0]=0$. Therefore, by Lemma 8.6, for every $a\neq 0$, $[\mathfrak{g}_{int},\mu]=[\mathfrak{g}_{int},\mu_0]$, hence μ is conjugate to μ_0 by an element in $\exp(ad(\mathfrak{g}_{int}))$. Let us now suppose $a=0$. Then, by the same arguments as above, we can assume either (i) $\mu=\xi_1\frac{\partial}{\partial x}+x\xi_1\xi_2\frac{\partial}{\partial\xi_1}+\tilde{h}$ for some $\tilde{h}\in\mathfrak{g}_1\cap\prod_{j\geq 3}S_j$, or (ii) $\mu=\xi_1\frac{\partial}{\partial x}+\tilde{h}$ for some $\tilde{h}\in\mathfrak{g}_1\cap\prod_{j\geq 3}S_j$. Notice that $\mu_1=\xi_1\frac{\partial}{\partial x}+x\xi_1\xi_2\frac{\partial}{\partial\xi_1}$ satisfies conditions (6.1) and (6.2) and $[\mathfrak{g}_{int},\mu]$ has codimension one in $[\mathfrak{g}_0,\mu]$. Therefore, by Lemma 8.6, in case (i) $[\mathfrak{g}_{int},\mu]=[\mathfrak{g}_{int},\mu_1]$, hence μ is conjugate to μ_1. Finally, in case (ii), $[\mathfrak{g}_{int},\mu]+\mathbb{F}\mu$ has at least codimension 2 in $\mathfrak{g}_1$, and this contradicts condition $[\mathfrak{g}_0,\mu]+\mathbb{F}\mu=\mathfrak{g}_1$, since $\mathfrak{g}_{int}$ has codimension 1 in $\mathfrak{g}_0$.

Now let $\mathfrak{g}=S(1,4)=\prod_{j\geq -1}\mathfrak{g}_j$ with the grading of type $(0|1,1,0,0)$. We have: $\mathfrak{g}_{-1}=\langle\frac{\partial}{\partial\xi_i}\mid i=1,2\rangle\otimes\mathbb{F}[[x]]\otimes\Lambda(\xi_3,\xi_4)$, $\mathfrak{g}_0=\langle\frac{\partial}{\partial x},\frac{\partial}{\partial\xi_k},\xi_j\frac{\partial}{\partial\xi_i}\mid k=3,4,i,j=1,2\rangle\otimes\mathbb{F}[[x]]\otimes\Lambda(\xi_3,\xi_4)$, $\mathfrak{g}_1=\langle\xi_i\frac{\partial}{\partial x},\xi_i\frac{\partial}{\partial\xi_k},\xi_1\xi_2\frac{\partial}{\partial\xi_i}\mid k=3,4,i=1,2\rangle\otimes\mathbb{F}[[x]]\otimes\Lambda(\xi_3,\xi_4)$.

Let $\mu\in(\mathfrak{g}_1)_{\bar{0}}$ satisfy properties (6.1) and (6.2). If $\mu\in\mathfrak{g}_1\cap\prod_{j\geq 1}S_j$, then the subalgebra of $\mathfrak{g}_0$ generated by $[\mathfrak{g}_{-1},\mu]$ is contained in $\prod_{j\geq 0}S_j$, hence it is properly contained in $\mathfrak{g}_0$. It follows that, up to a linear change of indeterminates, we may assume $\mu=\xi_1\frac{\partial}{\partial\xi_3}+\xi_2\frac{\partial}{\partial\xi_4}+h$ for some $h\in\mathfrak{g}_1\cap\prod_{j\geq 1}S_j$. We will show that, up to conjugation by elements in $\exp(ad(\mathfrak{g}_{int}))$, we may assume either $h=\alpha\xi_1\xi_4\frac{\partial}{\partial x}+\xi_1\xi_3\frac{\partial}{\partial x}+\varphi$ for some $\alpha\in\mathbb{F}$, $\alpha\neq 0$ and some $\varphi\in\prod_{j\geq 2}S_j$, or $h=\xi_1\xi_3\frac{\partial}{\partial x}+x\xi_2\xi_4\frac{\partial}{\partial x}-\xi_2\xi_3\xi_4\frac{\partial}{\partial\xi_3}+\psi$ for some $\psi\in\prod_{j\geq 3}S_j$. Indeed, let $\mu=\xi_1\frac{\partial}{\partial\xi_3}+\xi_2\frac{\partial}{\partial\xi_4}+\alpha\xi_1\xi_4\frac{\partial}{\partial x}+\beta\xi_1\xi_3\frac{\partial}{\partial x}+\gamma\xi_2\xi_4\frac{\partial}{\partial x}+\delta\xi_2\xi_3\frac{\partial}{\partial x}+Ax\xi_1\frac{\partial}{\partial\xi_3}+Bx\xi_2\frac{\partial}{\partial\xi_3}+Cx\xi_1\frac{\partial}{\partial\xi_4}+Dx\xi_2\frac{\partial}{\partial\xi_4}+h_1$ for some $\alpha,\beta,\gamma,\delta,A,B,C,D\in\mathbb{F}$ and some $h_1\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. Then $\exp(ad(-Ax\xi_1\frac{\partial}{\partial\xi_1}))(\mu)=\mu-Ax\xi_1\frac{\partial}{\partial\xi_3}+h_1'$ for some $h_1'\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. It follows that we may assume $A=0$. Then $\exp(ad(-Bx\xi_2\frac{\partial}{\partial\xi_1}))(\mu)=\mu-Bx\xi_2\frac{\partial}{\partial\xi_3}+h_1''$ for some $h_1''\in\mathfrak{g}_1\cap\prod_{j\geq 2}S_j$. It follows that we may assume $B=0$. By similar arguments we may assume $C=D=0$. Then $\exp(ad(-\delta\xi_3\xi_4\frac{\partial}{\partial x}))(\mu)=\mu+\delta\xi_1\xi_4\frac{\partial}{\partial x}-\delta\xi_2\xi_3\frac{\partial}{\partial x}$ hence we may assume $\delta=0$, i.e., $\mu=\xi_1\frac{\partial}{\partial\xi_3}+\xi_2\frac{\partial}{\partial\xi_4}+\alpha\xi_1\xi_4\frac{\partial}{\partial x}+\beta\xi_1\xi_3\frac{\partial}{\partial x}+\gamma\xi_2\xi_4\frac{\partial}{\partial x}+h_1$. Notice that, since $[\mathfrak{g}_{-1},\mu]$ generates $\mathfrak{g}_0$ and $\frac{\partial}{\partial x}$ lies in $\mathfrak{g}_0$, we necessarily have $(\alpha,\beta,\gamma)\neq(0,0,0)$.

By direct computation one shows that $\mu_{\alpha,\beta,\gamma}=\xi_1\frac{\partial}{\partial\xi_3}+\xi_2\frac{\partial}{\partial\xi_4}+\alpha\xi_1\xi_4\frac{\partial}{\partial x}+\beta\xi_1\xi_3\frac{\partial}{\partial x}+\gamma\xi_2\xi_4\frac{\partial}{\partial x}$ satisfies conditions (6.1) and (6.2) for every $\alpha,\beta,\gamma\in\mathbb{F}$ such that $\alpha\neq 0$ or $\alpha=0$ and $\beta\neq 0\neq\gamma$. Besides, $[\mathfrak{g}_0,\mu_{\alpha,\beta,\gamma}]=[\mathfrak{g}_{int},\mu_{\alpha,\beta,\gamma}]$, since $[\frac{\partial}{\partial x},\mu_{\alpha,\beta,\gamma}]=0$. It follows that the elements $\mu_{\alpha,\beta,\gamma}$, with $\alpha,\beta,\gamma\in\mathbb{F}$ such that $\alpha\neq 0$ or $\alpha=0$ and $\beta\neq 0\neq\gamma$, are conjugate to each other by automorphisms in $exp(ad(\mathfrak{g}_{int}))$. Therefore, if $\alpha\neq 0$ or $\alpha=0$ and $\beta\neq 0\neq\gamma$, by Lemma 8.6, $[\mathfrak{g}_{int},\mu]=[\mathfrak{g}_{int},\mu_{\alpha,\beta,\gamma}]=[\mathfrak{g}_{int},\mu_{a,1,0}]$ for $a\neq 0$, hence, under these hypotheses, μ is conjugate to $\mu_{a,1,0}$ by an element in $\exp(ad(\mathfrak{g}_{int}))$.

Now suppose $\alpha = 0 = \gamma$ and $\beta \neq 0$ (the case $\alpha = 0 = \beta$ and $\gamma \neq 0$ is equivalent to this up to a linear change of indeterminates). Then we may assume $\mu = \xi_1 \frac{\partial}{\partial \xi_3} + \xi_2 \frac{\partial}{\partial \xi_4} + \xi_1\xi_3 \frac{\partial}{\partial x} + q$ for some $q \in \prod_{j \geq 2} S_j$.

Notice that $\tilde{\mu}_{a,b,c} = \xi_1 \frac{\partial}{\partial \xi_3} + \xi_2 \frac{\partial}{\partial \xi_4} + \xi_1\xi_3 \frac{\partial}{\partial x} + ax\xi_2\xi_4 \frac{\partial}{\partial x} + b\xi_1\xi_2\xi_4 \frac{\partial}{\partial \xi_1} + c\xi_2\xi_3\xi_4 \frac{\partial}{\partial \xi_3}$ satisfies conditions (6.1) and (6.2) for every $a, b, c \in \mathbb{F}$ such that $a \neq 0$ and $a - b + c = 0$ but in this case $[\mathfrak{g}_{int}, \tilde{\mu}_{a,b,c}] + \mathbb{F}\tilde{\mu}_{a,b,c}$ has codimension one in $[\mathfrak{g}_0, \tilde{\mu}_{a,b,c}] + \mathbb{F}\tilde{\mu}_{a,b,c}$. More precisely, $[\mathfrak{g}_{int}, \tilde{\mu}_{a,b,c}] \oplus \mathbb{F}\xi_2\xi_4 \frac{\partial}{\partial x} = [\mathfrak{g}_0, \tilde{\mu}_{a,b,c}]$. It follows that, for $a, b, c \in \mathbb{F}$ such that $a \neq 0$ and $a - b + c = 0$, the elements $\tilde{\mu}_{a,b,c}$ are conjugate to each other by automorphisms in $exp(ad(\mathfrak{g}_{int}))$.

One verifies that the vector field $\xi_2\xi_4 \frac{\partial}{\partial x}$ lies in $[\mathfrak{g}_0, \mu] + \mathbb{F}\mu$ if and only if $q = ax\xi_2\xi_4 \frac{\partial}{\partial x} + b\xi_1\xi_2\xi_4 \frac{\partial}{\partial \xi_1} + c\xi_2\xi_3\xi_4 \frac{\partial}{\partial \xi_3} + h + t$ for some $a, b, c \in \mathbb{F}$, with $a \neq 0$ and $a - b + c = 0$, some $t \in \prod_{j \geq 3} S_j$ and some $h \in \langle x\xi_1\xi_3 \frac{\partial}{\partial x}, x\xi_1\xi_4 \frac{\partial}{\partial x}, x\xi_2\xi_3 \frac{\partial}{\partial x}, \xi_1\xi_3\xi_4 \frac{\partial}{\partial x}, x^2\xi_i \frac{\partial}{\partial \xi_4}, \xi_1\xi_2\xi_k \frac{\partial}{\partial \xi_i} \mid i = 1, 2, k = 3, 4\rangle$. It follows that $[\mathfrak{g}_{int}, \mu] + \mathbb{F}\mu$ consists of vector fields in $\mathfrak{g}_1$ not containing the element $\xi_2\xi_4 \frac{\partial}{\partial x}$. Therefore μ is conjugate to $\tilde{\mu}_{a,b,c}$ for $a \neq 0$ and $a - b + c = 0$, i.e., to $\tilde{\mu}_{1,0,-1}$.

Let us now consider $\mathfrak{g} = W(2,1) = \prod_{j \geq -1} \mathfrak{g}_j$ with the grading of type $(1,0|0)$. We have: $\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial x_1} \rangle \otimes A$, $\mathfrak{g}_0 = \langle x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \xi} \rangle \otimes A$, $\mathfrak{g}_1 = \langle x_1^2 \frac{\partial}{\partial x_1}, x_1 \frac{\partial}{\partial x_2}, x_1 \frac{\partial}{\partial \xi} \rangle \otimes A$, where $A = \mathbb{F}[[x_2]] \otimes \Lambda(\xi)$.

Let $\mu \in (\mathfrak{g}_1)_{\bar{1}}$ satisfy properties (6.1) and (6.2). If $\mu \in \mathfrak{g}_1 \cap \prod_{j \geq 1} S_j$, then the subalgebra of $\mathfrak{g}_0$ generated by $[\mathfrak{g}_{-1}, \mu]$ is contained in S_0, hence it is properly contained in $\mathfrak{g}_0$. It follows that, up to a linear change of indeterminates, we may assume $\mu = x_1 \frac{\partial}{\partial \xi} + h$ for some $h \in \prod_{j \geq 1} S_j$. Let $\mu = x_1 \frac{\partial}{\partial \xi} + ax_1\xi \frac{\partial}{\partial x_2} + bx_1x_2 \frac{\partial}{\partial \xi} + h'$, for some $a, b \in \mathbb{F}$ and some $h' \in \prod_{j \geq 2} S_j$. Since the subalgebra generated by $[\mathfrak{g}_{-1}, \mu]$ contains $\frac{\partial}{\partial x_2}$, we have $a \neq 0$. Besides, $\exp(ad(-bx_1x_2 \frac{\partial}{\partial x_1}))(\mu) = \mu - bx_1x_2 \frac{\partial}{\partial \xi} + h''$, for some $h'' \in \prod_{j \geq 2} S_j$. Therefore we can assume $\mu = x_1 \frac{\partial}{\partial \xi} + x_1\xi \frac{\partial}{\partial x_2} + \tilde{h}$ for some $\tilde{h} \in \prod_{j \geq 2} S_j$. By direct computation one shows that if $\mu_0 = x_1 \frac{\partial}{\partial \xi} + x_1\xi \frac{\partial}{\partial x_2}$, then $[\mathfrak{g}_0, \mu_0] = [\mathfrak{g}_{int}, \mu_0] = \mathfrak{g}_1$. By Lemma 8.6, μ is conjugate to μ_0. Note that the subalgebra of $\mathfrak{g}_0$ generated by $[\mathfrak{g}_{-1}, \mu_0]$ is properly contained in $\mathfrak{g}_0$ since μ_0 and all vector fields in $\mathfrak{g}_{-1}$ have zero divergence. Let $\mu_1 = \mu_0 + x_1^2\xi \frac{\partial}{\partial x_1}$. Then $[\mathfrak{g}_{int}, \mu_1] = [\mathfrak{g}_{int}, \mu_0]$ by Lemma 8.6, and a direct computation shows that $[\mathfrak{g}_{-1}, \mu_0]$ generates $\mathfrak{g}_0$. It follows that μ is conjugate to μ_1.

By similar arguments one proves cases 6.–21. $\square$

Remark 8.9 We recall that the Lie superalgebras $S(1,2)$ and $SKO(2,3;1)$ have an sl_2-copy $\mathfrak{a}$ of outer derivations. We shall denote by e, f, h the standard basis of $\mathfrak{a}$ defined in [8, Lemma 5.9] and [3, Remark 4.15], respectively. Besides, $DerSHO(3,3) = SHO(3,3) + \mathfrak{a}$ with $\mathfrak{a} \cong gl_2$ [3, Proposition 1.8]. The subalgebra $\mathfrak{a}$ of outer derivations of $SHO(3,3)$ is generated by the Euler operator E and by a copy of sl_2 with Chevalley basis $\{e, h.f\}$ where

$$e = ad(\xi_1\xi_3 \frac{\partial}{\partial x_2} - \xi_2\xi_3 \frac{\partial}{\partial x_1} - \xi_1\xi_2 \frac{\partial}{\partial x_3}) \quad \text{and} \quad h = -ad \sum_{i=1}^{3} \xi_i \frac{\partial}{\partial \xi_i}.$$

(Note that the formula for h, given in [3, Remark 2.37], is incorrect.) In order to define f, consider $SHO(3,3)$ with its principal grading, i.e., the grading of type $(1,1,1|1,1,1)$. In this grading, for every $j > 1$, $SHO(3,3)_j = SHO(3,3)_1^j$, therefore it is sufficient to define f on the local part $SHO(3,3)_{-1} \oplus SHO(3,3)_0 \oplus SHO(3,3)_1$ of $SHO(3,3)$. One has: $f(\xi_1\xi_2) = x_3$, $f(\xi_1\xi_3) = -x_2$, $f(\xi_2\xi_3) = x_1$; $f(x_1\xi_2\xi_3) = \frac{1}{2}x_1^2$, $f(x_2\xi_1\xi_3) = -\frac{1}{2}x_2^2$, $f(x_3\xi_1\xi_2) = \frac{1}{2}x_3^2$, $f(x_1\xi_1\xi_2 - x_3\xi_3\xi_2) = x_1x_3$, $f(x_2\xi_2\xi_1 - x_3\xi_3\xi_1) = -x_2x_3$, $f(x_1\xi_1\xi_3 - x_2\xi_2\xi_3) = -x_1x_2$, and $f = 0$ elsewhere on $SHO(3,3)_{-1} \oplus SHO(3,3)_0 \oplus SHO(3,3)_1$.

Theorem 8.10 *Let $L = S \rtimes \mathbb{F}d$, where S is a simple linearly compact Lie superalgebra and d is an even outer derivation of S. Then an even admissible grading of L such that d has degree 1, is either short or isomorphic to one of the following:*

1. $S = SHO(3,3)$, $(1,0,0|0,1,1)$, $d = e$;

2. $S = SKO(2,3;1)$, $(1,0|0,1,1)$, $d = e$.

Proof. Let $L = \prod_{j \geq -1} L_j$ be a $\mathbb{Z}$-grading of L of depth 1. According to [10, Proposition 5.1.2], [12] and [3, Proposition 1.8], one of the following possibilities may occur:

i) $S = A(1,1)$, $d = D_1$ (see [10, Proposition 5.1.2(e)]);

ii) $S = H(0,n)$ for some even $n > 5$ and $d = \xi_1 \dots \xi_n$;

iii) $S = S(1,n)$ for some even $n \geq 4$ and $d = \xi_1 \dots \xi_n \frac{\partial}{\partial x}$;

iv) $S = SKO(m, m+1; (m-2)/m)$ for some odd $m \geq 3$ and $d = \xi_1 \dots \xi_m$;

v) $S = SKO(m, m+1; 1)$ for some even $m \geq 4$ and $d = \tau\xi_1 \dots \xi_m$;

vi) $S = SHO(m,m)$ for some odd $m \geq 5$ and $d = \xi_1 \dots \xi_m$;

vii) $S = S(1,2)$ and $d = e$;

viii) $S = SHO(3,3)$ and $d = e$;

ix) $S = SKO(2,3;1)$ and $d = e$.

Let us analyze all cases $i) - ix)$. From [10, Proposition 5.1.2(e)] one deduces that if $S = A(1,1)$ and $L = S \rtimes \mathbb{F}D_1$, then any $\mathbb{Z}$-grading of L of depth 1, such that D_1 has degree 1, is short.

Let $S = H(0,n)$ for some even $n > 5$, say $n = 2t$. By Proposition 8.1, the gradings of depth 1 of L are, up to isomorphism, the gradings of type $(|1,\dots,1)$ and the grading of type $(|1,\dots,1,0,\dots,0)$ with t 1's and t 0's. In the grading of type $(|1,\dots,1)$, d has degree $n-2 > 1$ (since $n \geq 5$) in the grading of type $(|1,\dots,1,0,\dots,0)$ it has degree $t-1 > 1$. Therefore if $L = H(0,n) \rtimes \mathbb{F}\xi_1 \dots \xi_n$ for some even $n > 5$, then L has no $\mathbb{Z}$-grading of depth one such that $\xi_1 \dots \xi_n$ has degree 1.

Let us consider case iii). By Proposition 8.1, the gradings of depth 1 of L such that d has degree 1, are, up to isomorphism, the gradings of type $(0|1,0,\dots,0)$ and $(1|1,1,0,\dots,0)$. The first one is short, therefore let us consider the second one. The 0th and 1st graded components of S in this grading have dimension $(n+6)2^{n-2}$ and $(3+3n)2^{n-2}$, respectively. Hence, by Remark 6.2, the grading of type $(1|1,1,0,\dots,0)$ is not even admissible.

Let us now consider case iv). By Proposition 8.1, L has, up to isomorphism, only one grading of depth 1 such that d has degree 1, i.e., the grading of type $(1,\dots,1,0,0|0,\dots,0,1,1,1)$. The 0th and 1st graded components of L in this grading are infinite-dimensional vector spaces of growth 2, and size $m2^{m-2}$ and $\frac{m^2+m-4}{2}2^{m-2}$, respectively. Hence, by Remark 6.2, the grading of type $(1,\dots,1,0,0|0,\dots,0,1,1,1)$ is not even admissible.

By similar arguments one can rule out cases v) and vi).

Let $L = S(1,2) + \mathbb{F}d$. Then, up to isomorphism, one can assume that the grading of L is either of type $(0|1,0)$ or of type $(1|1,1)$. The first grading is short, thus let us consider the second one. The 0th and 1st graded components of L in this grading are finite-dimensional vector spaces of

dimension 8 and 9, respectively. One shows that for every choice of $\mu \in (L_1)_{\bar{0}}$, the centralizer of μ in L_0 has positive dimension, hence μ cannot satisfy property (6.1) of Definition 6.1.

Let $S = SHO(3,3)$. By Proposition 8.1, the gradings of depth 1 of L such that e has degree 1, are, up to isomorphism, the gradings of type $(1,1,1|1,1,1)$ and $(1,0,0|0,1,1)$. The 0th and 1st graded component of L in the grading of type $(1,1,1|1,1,1)$ have dimension 17 and 32, respectively, hence this grading is not even admissible. On the contrary the grading of type $(1,0,0|0,1,1)$ is even admissible since one can check, by direct calculations, that it satisfies Definition 6.1 with $\mu = e + x_1\xi_2$. Notice that the gradings of type $(1,1,1|1,1,1)$ and $(1,0,0|0,1,1)$ of $S + \mathbb{F}e$ are isomorphic to the gradings of type $(1,1,1|0,0,0)$ and $(1,0,0|-1,0,0)$, respectively, via the map $\exp(e)\exp(-f)\exp(e)$ (see also [3, Remark 2.38]).

Similarly, if $S = SKO(2,3;1)$, then the Lie superalgebra $L = SKO(2,3;1) + \mathbb{F}e$ has one even admissible grading, namely the grading of type $(1,0|0,1,1)$. In this case $\mu = e + x_1\xi_2$. $\square$

Theorem 8.11 *Let $L = S \rtimes \mathbb{F}d$ as in the statement of Theorem 8.10 and let $L = \prod_{j\geq -1} L_j$ be an even admissible grading of L, of height strictly greater than one, such that d has degree one. Then a complete list, up to conjugation by $\exp(ad(L_{int}))$, of elements $\mu \in L_1$ satisfying properties (6.1) and (6.2), is as follows:*

1. $S = SHO(3,3)$ *with the grading of type* $(1,0,0|0,1,1)$*:* $\mu = e + x_1\xi_2$ *and* $[L_0,\mu] + \mathbb{F}\mu = L_1$*;*
2. $S = SKO(2,3;1)$ *with the grading of type* $(1,0|0,1,1)$*:* $\mu = e + x_1\xi_2$ *and* $[L_0,\mu] + \mathbb{F}\mu = L_1$*.*

Proof. We will denote by $S = \prod_j S_j$ the principal grading of S and argue as in the proof of Theorem 8.8. Let us first consider the case $S = SHO(3,3)$ with the grading of type $(1,0,0|0,1,1)$. We have: $L_{-1} = A/\mathbb{F}1$, with $A = \mathbb{F}[[x_2,x_3]] \otimes \Lambda(\xi_1)$; $L_0 = \{f \in \langle x_1,\xi_2,\xi_3\rangle \otimes A \mid \Delta(f) = 0\}$; $L_1 = \{f \in \langle x_1^2, x_1\xi_2, x_1\xi_3, \xi_2\xi_3\rangle \otimes A \mid \Delta(f) = 0\}$. Let $\mu \in (L_1)_{\bar{0}}$ satisfy properties (6.1) and (6.2). Then we may assume, up to a linear change of indeterminates, that $\mu = x_1\xi_2 + h$ for some $h \in \prod_{j\geq 1} S_j$. Indeed, if μ lies in $L_1 \cap \prod_{j\geq 1} S_j$, then $[L_{-1},\mu]$ is contained in $\prod_{j\geq 0} S_j$ hence it cannot generate L_0, contradicting property (6.2). Up to automorphism in $\exp(ad(L_{int}))$ we may thus assume $\mu = x_1\xi_2 + \alpha e + h'$ for some $h' \in \prod_{j\geq 2} S_j$ and some $\alpha \in \mathbb{F}$. By Theorem 8.3*i*), we have $\alpha \neq 0$, hence we may assume $\alpha = 1$. One checks, by direct computations, that the element $\mu_0 = x_1\xi_2 + e$ satisfies properties (6.1) and (6.2). Besides $[L_0,\mu_0] = [L_{int},\mu_0]$, since $[\xi_i,\mu_0] = 0$ for $i = 1,2$. Then statement 1. follows from Lemma 8.6.

Now let $S = SKO(2,3;1)$ with the grading of type $(1,0|0,1,1)$. We have: $L_{-1} = \mathbb{F}[[x_2]] \otimes \Lambda(\xi_1) =: A$, $L_0 = \{f \in \langle x_1,\xi_2,\tau\rangle \otimes A \mid div_1(f) = 0\}$, $L_1 = \{f \in \langle x_1^2, x_1\xi_2, x_1\tau, \xi_2\tau\rangle \otimes A \mid div_1(f) = 0\}$. Let $\mu \in (L_0)_{\bar{0}}$ satisfy properties (6.1), (6.2). Then $\mu = z_0 + z_1 + h$ for some $h \in \prod_{j\geq 2} S_j$ and some $z_i \in S_i$, $i = 1,2$, such that $z_0 + z_1 \neq 0$, since if μ lies in $\prod_{j\geq 2} S_j$, then $[L_{-1},\mu] \subset L_0 \cap \prod_{j\geq 0} S_j$ contradicting (6.2). Suppose that $\mu = x_1\xi_2 + h'$ for some $h' \in \prod_{j\geq 1} S_j$. Then, we may assume, up to automorphisms, $\mu = x_1\xi_2 + \alpha e + h$ for some $h \in \prod_{j\geq 3} S_j$ and some $\alpha \in \mathbb{F}$. Besides, by Theorem 8.3*i*), we have $\alpha \neq 0$, hence we may suppose $\alpha = 1$. A direct computation shows that the element $\mu_0 = x_1\xi_2 + e$ satisfies (6.1) and (6.2); besides, $[L_0,\mu] = [L_{int},\mu]$ since $[\xi_2,\mu] = 0$. By Lemma 8.6 μ is conjugate to μ_0.

Now suppose that $z_0 = 0$, i.e., $\mu = Ax_1^2\xi_1 + Bx_1\tau + Cx_1x_2\xi_2 + h$ for some $h \in \prod_{j\geq 2} S_j$ and some $A,B,C \in \mathbb{F}$ such that $2A - B + C = 0$. Notice that if $B = 0$, then $[L_{-1},\mu] \subset L_0 \cap \prod_{j\geq 0} S_j$ contradicting (6.2), hence we may assume $B = 1$. It follows that, up to automorphisms, we may assume $\mu = x_1\tau + Ax_1^2\xi_1 + Cx_1x_2\xi_2 + \alpha e + h'$ for some $h' \in \prod_{j\geq 3} S_j$ and some $\alpha, A, C \in \mathbb{F}$ such that

$2A+C-1=0$. By Theorem 8.3i) we have $\alpha \neq 0$. Then one checks that $L_1 \cap S_1$ is not contained in $[L_0, \mu] + \mathbb{F}\mu$, hence contradicting property (6.1). $\square$

Proposition 8.12 Let S be a simple linearly compact Lie superalgebra.

a) Let $L = S + \mathbb{F}\mu + \mathbb{F}[\mu, \mu]$, where μ is an odd outer derivation of S such that $[\mu, \mu] \neq 0$. Then there is no admissible grading of L such that μ has degree 1.

b) If $L = S \otimes \Lambda(1) + \mathbb{F}\mu + \mathbb{F}d$, where d is an even outer derivation of S and $\mu = d \otimes \xi + 1 \otimes d/d\xi$, then there exists no $\mathbb{Z}$-grading of L of depth 1 such that μ has degree 1.

Proof. a) From the description of outer derivations of simple linearly compact Lie superalgebras given in [10, Proposition 5.1.2], [12] and [3, Proposition 1.8], we deduce that $L = S \rtimes \mathbb{F}d$, where the following possibilities for S and d may occur, up to isomorphism:

i) $S = q(n)$ and $d = D$ (see [10, Proposition 5.1.2(c)] for the definition of D);

ii) $S = H(0, n)$ for some odd $n \geq 5$ and $d = \xi_1 \dots \xi_n$;

iii) $S = S(1, n)$ for some odd $n \geq 3$ and $d = \xi_1 \dots \xi_n \frac{\partial}{\partial x}$;

iv) $S = SKO(m, m+1; (m-2)/m)$ for some even $m \geq 2$ and $d = \xi_1 \dots \xi_m$;

v) $S = SKO(m, m+1; 1)$ for some odd $m \geq 3$ and $d = \tau\xi_1 \dots \xi_m$;

vi) $S = SHO(m, m)$ for some even $m \geq 4$ and $d = \xi_1 \dots \xi_m$.

In all cases $[d, d] = 0$, hence $\mu = d + z$ for some odd inner derivation $z \neq 0$, and d and z have degree 1. In case i) it follows immediately from the definition that every $\mathbb{Z}$-grading of depth 1 of L is short, hence there is no such μ. Let us consider case iii). By Proposition 8.1, the gradings of depth 1 of L such that d has degree 1, are, up to isomorphism, the gradings of type $(0|1, 0, \dots, 0)$ and $(1|1, 1, 0, \dots, 0)$. The first one is short, hence there is no odd derivation μ satisfying the hypotheses. Let us now consider the grading of type $(1|1, 1, 0, \dots, 0)$. The 0th and 1st graded components of S in this grading have dimension $(n+6)2^{n-2}$ and $(3+3n)2^{n-2}$, respectively. Hence, by Remark 6.2, the grading of type $(1|1, 1, 0, \dots, 0)$ is not admissible. Similarly one rules out cases ii), v), vi) and iv) unless $S = SKO(2, 3; 0)$ with the grading of type $(0, 0|1, 1, 1)$. Since this grading is short we get statement a).

Now let L, μ and d be as in b) and consider a $\mathbb{Z}$-grading of depth 1 of L such that μ has degree 1. Then ξ has degree -1, hence $S = \prod_{j \geq 0} S_j$, and d has degree 2. Since S is simple it follows that $S = S_0$, therefore there is no outer derivation of S of degree 2. Statement b) follows. $\square$

Remark 8.13 Let $(\mathfrak{g}, \mu)$ be one of the pairs listed in Tables 1 and 2, where $\mathfrak{g} = \prod_{i \geq -1} \mathfrak{g}_i$ is a simple linearly compact Lie superalgebra with an admissible grading, and $\mu \in \mathfrak{g}_1$ satisfies Definition 6.1. According to Proposition 6.3, $J(\mathfrak{g}, \mu) = \mathfrak{g}_{-1}$ with product $f \circ g = [[\mu, f], g]$. We hence get the following corresponding list of rigid superalgebras (constructed in Sections 4 and 5), where the bar over J and μ means parity reversal, described in Remark 1.4:

1. $\mathfrak{g} = W(0, 3)$ with the grading of type $(|1, 1, 0)$, $J(\mathfrak{g}, \mu) = JW_{0,4}$;

2. $\mathfrak{g} = W(0, 4)$ with the grading of type $(|1, 1, 0, 0)$, $J(\mathfrak{g}, \mu) = JW_{0,8}$;

3. $\mathfrak{g}=S(0,4)$ with the grading of type $(|1,1,0,0)$, $J(\mathfrak{g},\mu)=JS_{0,8}$;

4. $\mathfrak{g}=S(0,5)$ with the grading of type $(|1,1,0,0,0)$, $J(\mathfrak{g},\mu)=JS_{0,16}$;

5. $\mathfrak{g}=S(1,4)$ with the grading of type $(0|1,1,0,0)$, $J(\mathfrak{g},\mu)=JS^{\alpha}_{1,8}$;

6. $\mathfrak{g}=S(2,0)$ with the grading of type $(1,1|)$, $J(\mathfrak{g},\mu)=JS_{0,2}$;

7. $\mathfrak{g}=S(2,0)$ with the grading of type $(1,0|)$, $J(\mathfrak{g},\mu)$ is the Beltrami algebra $JS_{1,1}$;

8. $\mathfrak{g}=W(1,1)$ with the grading of type $(1|0)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LW_{0,2}$;

9. $\mathfrak{g}=W(1,2)$ with the grading of type $(0|1,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LW^{\alpha}_{1,2}$;

10. $\mathfrak{g}=W(2,1)$ with the grading of type $(1,0|0)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LW_{1,2}$;

11. $\mathfrak{g}=W(2,2)$ with the grading of type $(0,0|1,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LW^{\alpha}_{2,2}$;

12. $\mathfrak{g}=S(2,2)$ with the grading of type $(0,0|1,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LS^{\alpha}_{2,2}$;

13. $\mathfrak{g}=S(1,3)$ with the grading of type $(0|1,1,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LS_{1,3}$;

14. $\mathfrak{g}=H(2,2)$ with the grading of type $(1,0|1,0)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LH^{\alpha}_{1,2}$;

15. $\mathfrak{g}=HO(2,2)$ with the grading of type $(1,0|0,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LHO_{1,2}$;

16. $\mathfrak{g}=HO(3,3)$ with the grading of type $(0,0,0|1,1,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LHO^{\alpha}_{3,1}$;

17. $\mathfrak{g}=HO(n+1,n+1)$, $n\geq 1$, with the grading of type $(0,\dots,0,1|0,\dots,0,-1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LP(n-1,n-1)$;

18. $\mathfrak{g}=SHO(3,3)$ with the grading of type $(1,0,0|0,1,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LSHO'_{2,2}$;

19. $\mathfrak{g}=SHO(4,4)$ with the grading of type $(0,0,0,0|1,1,1,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LSHO^{\alpha}_{4,1}$;

20. $\mathfrak{g}=SHO(n+1,n+1)$, $n\geq 2$, with the grading of type $(0,\dots,0,1|0,\dots,0,-1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LSHO_{n,2^{n-1}}$;

21. $\mathfrak{g}=KO(n+1,n+2)$, $n\geq 1$, with the grading of type $(0,\dots,0,1|0,\dots,0,-1,0)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LP(n-1,n)$;

22. $\mathfrak{g}=KO(2,3)$ with the grading of type $(0,0|1,1,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LKO_{2,1}$;

23. $\mathfrak{g}=SKO(n,n+1;\beta)$, $n\geq 2$, $\beta\neq\frac{4}{n}$, with the grading of type $(0,\dots,0,1|0,\dots,0,\,-1,0)$, $\mu=x_n+x_n\xi_{n-1}\tau-\frac{2-n\beta}{2}x_n^2\xi_{n-1}\xi_n$, $J(\bar{\mathfrak{g}},\bar{\mu})=LSKO_{n,2^n}$;

24. $\mathfrak{g}=SKO(2,3;\beta)$, $\beta\neq 0,1,2+\frac{1}{b},2+\frac{2}{b}$, $\forall b\in\mathbb{Z}_{>0}$, with the grading of type $(0,1|0,-1,0)$, $\mu=x_2+x_2^2\xi_2\tau-2x_1x_2\xi_1\tau+(3-2\beta)x_1x_2^2\xi_1\xi_2$, $J(\bar{\mathfrak{g}},\bar{\mu})=LSKO'_{1,2}$;

25. $\mathfrak{g}=SKO(3,4;1)$, with the grading of type $(0,0,1|0,0,-1,0)$, $\mu=x_3+x_3\xi_1\xi_2+x_3\xi_2\tau+\frac{1}{2}x_3^2\xi_2\xi_3$, $J(\bar{\mathfrak{g}},\bar{\mu})=LSKO'_{2,4}$;

26. $\mathfrak{g}=SKO(3,4;\beta)$, with the grading of type $(0,0,0|1,1,1,1)$, $J(\bar{\mathfrak{g}},\bar{\mu})=LSKO^{\alpha}_{3,1}$.

Similarly, we have the following remark.

Remark 8.14 Let (L,μ) be one of the pairs listed in Theorem 8.11. Then the corresponding rigid superalgebras are as follows:

1. $L = SHO(3,3) \rtimes \mathbb{F}e$, $J(L,\mu) = JSHO_{2,2}$;

2. $L = SKO(2,3;1) \rtimes \mathbb{F}e$, $J(L,\mu) = JSKO_{1,2}$.

Remark 8.15 Note that, since the Lie superalgebras $HO(2,2)$ and $SKO(2,3;0)$ are isomorphic [3, §0], the anti-commutative superalgebras $LSKO_{1,2}$ and $LHO_{1,2}$ are isomorphic.

9 Proofs of Theorems 4.5 and 5.25 and a corollary

Proof of Theorem 4.5 Let J be a simple linearly compact rigid superalgebra with product μ. Then, by Proposition 7.2, $Lie(J,\mu)$ is a linearly compact Lie superalgebra with an even admissible $\mathbb{Z}$-grading $Lie(J,\mu) = \prod_{k\geq -1} Lie_k(J)$, where $Lie_{-1}(J) = J$, $Lie_0(J) = Str(J,\mu)$ and $Lie_1(J) = R(J,\mu)$. By Theorem 7.3 either $Lie(J,\mu)$ is simple or $Lie(J,\mu) = S \rtimes \mathbb{F}\mu$, where S is a simple Lie superalgebra and μ is an even outer derivation of S lying in $Lie_1(J)$. The statement then follows from Theorem 8.3i), Remark 6.4, Theorems 8.8 and 8.11, Proposition 8.5, and Remarks 8.13, 8.14. □

Proof of Theorem 5.25 Let J be a simple linearly compact rigid odd type superalgebra with product μ. Then, by Proposition 7.2, $Lie(J,\mu)$ is a linearly compact Lie superalgebra with an odd admissible $\mathbb{Z}$-grading $Lie(J,\mu) = \prod_{k\geq -1} Lie_k(J)$, where $Lie_{-1}(J) = J$, $Lie_0(J) = Str(J,\mu)$ and $Lie_1(J) = R(J,\mu)$. By Theorem 7.3 and Proposition 8.12, either $Lie(J,\mu) \cong \xi\mathfrak{a} + \mathfrak{a} + d/d\xi$, where $\mathfrak{a}$ is a simple Lie superalgebra and ξ is an odd indeterminate, hence $(\bar{J},\bar{\mu})$ is a Lie superalgebra, or else $Lie(J,\mu)$ is simple. The statement then follows from Theorem 8.3ii), Proposition 6.5, Theorem 8.8, Proposition 8.5 and Remark 8.13. □

Proposition 9.1 *Let P be a linearly compact odd generalized Poisson superalgebra. If P is simple, then the rigid odd type superalgebra OJP, constructed in Example 5.1, is simple.*

Proof. We will divide the proof into three steps: let I be an ideal of OJP, first we will show that if I contains a non-zero element of the form $\eta f a$ for some $f \in P$ and $a \in \mathbb{F}[[x]]$, then $I = OJP$; secondly we will show that if I contains a non-zero element of the form fa for some $f \in P$ and $a \in \mathbb{F}[[x]]$, then $I = OJP$; finally we will show that every non-zero ideal of OJP necessarily contains either an element of the form $\eta f a$ or an element of the form fa for some $0 \neq f \in P$ and some $0 \neq a \in \mathbb{F}[[x]]$.

STEP 1. Suppose that I contains a non-zero element $\eta f a$ for some $f \in P$ and some $a \in \mathbb{F}[[x]]$. Then $\eta \circ \eta f a = -\eta f a' \in I$, hence, either a is a polynomial and $\eta f \in I$, or a is an infinite series and we may assume that it is invertible, i.e., $a = \sum_{k\geq 0} \alpha_k x^k$ with $\alpha_0 \neq 0$. We have: $\eta x^r \circ \eta f a = \eta f(r x^{r-1} a - x^r a') = \eta f(r\alpha_0 x^{r-1} + \text{higher order terms})$. It follows that in the limit we can cancel out all terms of $\eta f a$ of degree in x greater than 0, i.e., $\eta f \in I$. Now, for $g, h \in P$ and $b \in \mathbb{F}[[x]]$, we have: $\eta g b \circ \eta f = (-1)^{p(g)} \eta g f b' \in I$ and $h \circ \eta f = \eta\{h,f\} - \frac{1}{2}(-1)^{p(h)} \eta D(h) f \in I$, hence $\eta\{h,f\} \in I$. Let $J = \{\varphi \in P \mid \eta\varphi \in I\}$. Then $J \neq 0$ since $f \in J$. Besides, we have just shown that J is an ideal of P with respect to both the associative and the Poisson product. By the

simplicity of P, $J = P$, i.e., $\eta P \subset I$. Now, for $a \in \mathbb{F}[[x]]$ and $g \in P$, we have: $\eta a \circ \eta g = \eta g a' \in I$, hence $\eta P[[x]] \subset I$. Likewise, $a \circ \eta g = -ga' + \eta D(g)(a - \frac{1}{2}xa') \in I$, hence $ga' \in I$, i.e., $P[[x]] \subset I$. It follows that $I = OJP$.

STEP 2. Suppose that I contains a non-zero element fa for some $f \in P$ and some $a \in \mathbb{F}[[x]]$. Then $1 \circ fa = -D(f)a + 2\eta fa \in I$, therefore if $D(f) = 0$, then $I = OJP$ by Step 1. Now suppose $D(f) \neq 0$. We have:

$$\underbrace{\eta \circ (\eta \circ \ldots (\eta}_{r\ times} \circ (1 \circ fa))) = -D(f)a^{(r)} + 2(-1)^r \eta f a^{(r)} \in I, \tag{9.1}$$

where we denoted by $a^{(r)}$ the r-th derivative of a with respect to x. Likewise,

$$\underbrace{\eta \circ (\eta \circ \ldots (\eta}_{r\ times} \circ fa)) = (-1)^r (fa^{(r)} - \frac{r}{2}\eta D(f) a^{(r-1)}) \in I. \tag{9.2}$$

If a is a polynomial then there exists some $s \in \mathbb{Z}_{>0}$ such that $a^{(s-1)} \neq 0$ and $a^{(s)} = 0$. By (9.2), $\eta D(f)a^{(s-1)} \in I$, hence $I = OJP$ by Step 1. If a is an infinite series then there exists $s \in \mathbb{Z}_{\geq 0}$ such that $a^{(s)}$ is invertible. For $r \in \mathbb{Z}_{\geq 0}$ we have: $x^r \circ (-D(f)a^{(s)} + 2(-1)^s \eta f a^{(s)}) = -2(-1)^s r f x^{r-1} a^{(s)} + (-2 - r + 2(-1)^s)\eta D(f) x^r a^{(s)} \in I$. Since $a^{(s)}$ is invertible, it follows that $f + \eta D(f) b \in I$ for some invertible $b \in \mathbb{F}[[x]]$. We have: $\eta x \circ (1 \circ (f + \eta D(f) b)) = 2\eta f \in I$, hence $I = OJP$ by Step 1.

STEP 3. Suppose that I contains an element $z = fa + \eta g b$ for some non-zero $f, g \in P$ and some non-zero $a, b \in \mathbb{F}[[x]]$. Then

$$\underbrace{\eta \circ (\eta \circ \ldots (\eta}_{r\ times} \circ (fa + \eta g b))) = -fa^{(r)} + \eta(\frac{r}{2} D(f) a^{(r-1)} - g b^{(r)}) \in I. \tag{9.3}$$

If a is a polynomial then there exists $s \in \mathbb{Z}_{>0}$ such that $a^{(s-1)} \neq 0$ and $a^{(s)} = 0$. Then $\eta(\frac{s}{2} D(f) a^{(s-1)} - g b^{(s)}) \in I$. If $\frac{s}{2} D(f) a^{(s-1)} - g b^{(s)} \neq 0$, then $\eta \circ \eta(\frac{s}{2} D(f) a^{(s-1)} - g b^{(s)}) = \eta g b^{(s+1)} \in I$. Therefore, either $b^{(s+1)} \neq 0$ and $I = OJP$ by Step 1, or $b^{(s)} = \beta \in \mathbb{F}$. Note that $a^{(s-1)} = \alpha \in \mathbb{F}$ since $a^{(s)} = 0$, i.e., $\eta(\frac{s}{2}\alpha D(f) + \beta g) \in I$ and, again, $I = OJP$ by Step 1. Now suppose that $\frac{s}{2} D(f) a^{(s-1)} - g b^{(s)} = 0$. Then, either $D(f) \neq 0$ or $D(f) = 0$. If $D(f) \neq 0$ then $g = \gamma D(f)$ and $a^{(s-1)} = \delta b^{(s)} \in \mathbb{F}$, for some $\gamma, \delta \in \mathbb{F}$, $\gamma \neq 0 \neq \delta$. By (9.3), $fa^{(s-1)} + \eta(-\frac{s-1}{2} D(f) a^{(s-2)} + g b^{(s-1)}) \in I$, hence $1 \circ (fa^{(s-1)} + \eta(-\frac{s-1}{2} D(f) a^{(s-2)} + g b^{(s-1)})) = -D(f) a^{(s-1)} + 2\eta f a^{(s-1)} \in I$, i.e., $-D(f) + 2\eta f \in I$. It follows that $\eta x \circ (-D(f) + 2\eta f) = 2\eta f \in I$ and $I = OJP$ by Step 1. If $D(f) = 0$, then either $g = 0$ or $g \neq 0$ and $b^{(s)} = 0$. If $g = 0$, then, by (9.3), $fa^{(s-1)} \in I$ hence $I = OJP$ by Step 2. If $g \neq 0$, then $b^{(s)} = 0$, i.e., $b^{(s-1)} \in \mathbb{F}$, i.e., we may assume, by (9.3) with $r = s - 1$, that $f + \eta g \in I$, hence either $g = 0$ and $I = OJP$ by Step 2, or $g \neq 0$ and $\eta x \circ (f + \eta g) = \eta g \in I$, hence $I = OJP$ by Step 1.

Now suppose that a is an infinite series, then, by (9.3), we may assume that $fa + \eta g b \in I$ for some invertible $a \in \mathbb{F}[[x]]$, some non-zero $f, g \in P$ and some non-zero $b \in \mathbb{F}[[x]]$. For $k \in \mathbb{Z}_{\geq 0}$, we have: $\eta x^k \circ (fa + \eta g b) = -fa' x^k + \frac{1-k}{2} \eta D(f) a x^k \in I$. It follows that in the limit we can cancel out all terms of fa of positive degree in x, and get $f + \eta(D(f)c + gd) \in I$ for some $c, d \in \mathbb{F}[[x]]$. We have: $1 \circ (f + \eta D(f) c + \eta g d) = -D(f) + 2\eta f + \eta D(g) d \in I$ and $\eta \circ (1 \circ (f + \eta D(f) c + \eta g d)) = -\eta D(g) d' \in I$. Therefore if $D(g) d' \neq 0$, then $I = OJP$ by Step 1, otherwise, either $D(g) = 0$ or $D(g) \neq 0$ and $d' = 0$. If $D(g) = 0$, then $-D(f) + 2\eta f \in I$, hence $\eta x \circ (-D(f) + 2\eta f) = 2\eta f \in I$, and $I = OJP$

by Step 1. If $D(g) \neq 0$ and $d' = 0$, i.e., $d \in \mathbb{F}$, then, either $d = 0$ and we proceed as above, or $d \neq 0$ and $\eta x \circ (-D(f) + 2\eta f + \eta D(g)d) = 2\eta f + \eta D(g)d \in I$. If $2f + D(g)d \neq 0$, $I = OJP$ by Step 1. If $2f + D(g)d = 0$, then $D(f) \in I$ hence if $D(f) \neq 0$ then $I = OJP$ by Step 2. Finally, if $2f + D(g)d = 0$ and $D(f) = 0$, then $f + \eta gd \in I$ hence $\eta x \circ (f + \eta gd) = \eta gd$, since $d' = 0$, and the result follows from Step 1. □

Corollary 9.2 *a) Let $(P, \{\cdot,\cdot\})$ be a simple linearly compact odd generalized Poisson superalgebra. Then the rigid odd type superalgebra OJP is isomorphic to one of the superalgebras $OJP(n,n)$, $OJP(n,n+1)$, constructed in Example 5.1.*

b) Any simple linearly compact odd generalized Poisson superalgebra is gauge equivalent to $PO(n,n)$ or $PO(n,n+1)$.

c) Any simple linearly compact odd Poisson superalgebra is isomorphic to $PO(n,n)$.

Proof. It will be convenient to talk here about odd type commutative superalgebras instead of anti-commutative superalgebras (related to each other by reversal of parity, as in Remark 1.4); we shall denote the former superalgebras by $OJX_{m,n}$ if the latter are denoted by $LX_{m,n}$. Let $(P, \{\cdot,\cdot\})$ be a simple linearly compact odd generalized Poisson superalgebra and suppose that P is non-trivial. We showed in Example 5.1 that OJP is a rigid odd type superalgebra, which is linearly compact by construction. By Proposition 9.1, OJP is simple hence it is isomorphic to one of the superalgebras listed in Theorem 5.25. We have $OJP = P[[x]] + \eta P[[x]]$ where η is an odd variable. It follows that OJP cannot be isomorphic to any of the following rigid superalgebras:

1) $OJW_{0,2}$ since it is finite-dimensional;

2) $OJW^{\alpha}_{1,2}$, $OJW^{\alpha}_{2,2}$, $OJS^{\alpha}_{2,2}$, $OJS_{1,3}$, $OJHO^{\alpha}_{3,1}$, $OJSHO^{\alpha}_{4,1}$, $OJKO_{2,1}$, $OJSKO^{\alpha}_{3,1}$, since these are completely odd superalgebras;

3) $OJW_{1,2}$, $OJSKO'_{1,2}$, (resp. $OJH^{\alpha}_{1,2}$, $OJHO_{1,2}$) since these are equal to $\mathbb{F}[[x]] \otimes \Lambda(1)$ (resp. to $(\mathbb{F}[[x]] \otimes \Lambda(1))/\mathbb{F}1$, $(\mathbb{F}[[x]] \otimes \Lambda(1))/\mathbb{F}\xi$);

4) $OJSHO'_{2,2}$, since $OJSHO'_{2,2} = (\mathbb{F}[[x_1,x_2]] \otimes \Lambda(1))/\mathbb{F}\xi$, hence if OJP were isomorphic to $OJSHO'_{2,2}$, P would be completely even.

Now let us consider the rigid odd type superalgebra $OJSHO_{n,2^{n-1}}$. We have: $OJSHO_{n,2^{n-1}} = \{f \in \mathcal{O}(x_1,\dots,x_n,\xi_1,\dots,\xi_n) \mid \Delta(f) = 0\}$. Suppose that there exists a super-space Q such that $OJSHO_{n,2^{n-1}} = Q[[x]] + \eta Q[[x]]$ for some odd element $\eta \in OJSHO_{n,2^{n-1}}$. Then every element $f \in \mathcal{O}(x_1,\dots,x_n)$ lies in $Q[[x]]$, since there exists no odd element g such that $f = \eta g$. In particular x_i lies in $Q[[x]]$ for every i. It follows that ηx_i lies in $\eta Q[[x]]$, hence $\Delta(\eta x_i) = 0$. But $\Delta(\eta x_i) = \Delta(\eta)x_i + \frac{\partial\eta}{\partial\xi_i} = \frac{\partial\eta}{\partial\xi_i}$ ($\eta \in OJSHO_{n,2^{n-1}}$, hence $\Delta(\eta) = 0$). It follows that η lies in $\mathcal{O}(x_1,\dots,x_n)$ and this is a contradiction since η is odd. Therefore OJP cannot be isomorphic to $OJSHO_{n,2^{n-1}}$ for any n.

Likewise, we will show that OJP cannot be isomorphic to the rigid superalgebra $OJSKO_{n,2^n}$ for any n. Suppose that $OJSKO_{n,2^n} = Q[[x]] + \eta Q[[x]]$ for some super-space Q and some odd element $\eta \in OJSKO_{n,2^n}$. According to Example 5.5, we have $\Delta(\eta) + (E - n\beta)\frac{\partial\eta}{\partial\tau} + (1-\beta)\frac{\partial\eta}{\partial\tau} = 0$. As above, x_i lies in $Q[[x]]$ for every i, hence $\eta x_i \in \eta Q[[x]]$. It follows that $\Delta(\eta x_i) + (E - n\beta)\frac{\partial(\eta x_i)}{\partial\tau} + (1-\beta)\frac{\partial(\eta x_i)}{\partial\tau} = 0$, i.e., $\frac{\partial\eta}{\partial\xi_i} = -x_i\frac{\partial\eta}{\partial\tau}$. Let $\eta = \tau\eta_1 + \eta_2$ for some $\eta_1, \eta_2 \in \mathcal{O}(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$.

Then, for $i=1,\ldots,n$, we have: $-\tau\frac{\partial\eta_1}{\partial\xi_i}+\frac{\partial\eta_2}{\partial\xi_i}=-x_i\eta_1$, hence $\frac{\partial\eta_1}{\partial\xi_i}=0\ \forall\ i$, i.e., $\eta_1\in\mathcal{O}(x_1,\ldots,x_n)$, and $\frac{\partial\eta_2}{\partial\xi_i}=-x_i\eta_1$. It follows that $\eta_2=-\eta_1\sum_{k=1}^n x_k\xi_k$, i.e., $\eta=\tau\eta_1-\eta_1\sum_{k=1}^n x_k\xi_k$. Therefore, for $i=1,\ldots,n$, the elements ξ_i's, lying in $OJSKO_{n,2^n}$, lie in $Q[[x]]$, since there exists no even element g such that $\xi_i=\eta g$. It follows that $\eta\xi_i\in\eta Q[[x]]$, hence $\Delta(\eta\xi_i)+(E-n\beta)(\eta_1\xi_i)+(1-\beta)(\eta_1\xi_i)=0$, i.e., $-\tau\frac{\partial\eta_1}{\partial x_i}+\frac{\partial\eta_1}{\partial x_i}(\sum_{k=1}^n x_k\xi_k)+2\eta_1\xi_i=0$. It follows that $\frac{\partial\eta_1}{\partial x_i}=0\ \forall\ i$, hence $\eta_1\in\mathbb{F}$, i.e., $\eta=\tau-\sum_{k=1}^n x_k\xi_k$. We have $\Delta(\eta)+(E-n\beta)(\frac{\partial\eta}{\partial\tau})+(1-\beta)(\frac{\partial\eta}{\partial\tau})=-n+1-(n+1)\beta=0$, which is impossible for every $\beta\neq\frac{1-n}{n+1}$. Besides, if $\beta=\frac{1-n}{n+1}$, $\Delta(\eta x_i)+(E-n\beta)(\frac{\partial(\eta x_i)}{\partial\tau})+(1-\beta)(\frac{\partial(\eta x_i)}{\partial\tau})=x_i$ and this is a contradiction since $\eta x_i\in OJSKO_{n,2^n}$. We conclude that OJP cannot be isomorphic to $OJSKO_{n,2^n}$ for any n. Similarly, OJP cannot be isomorphic to $OJSKO'_{2,4}$. Statement a) follows.

In order to prove b), let us consider a simple odd generalized Poisson superalgebra P and the corresponding rigid odd type superalgebra $OJP=P[[x]]+\eta P[[x]]$ with product (5.1). By a), OJP is isomorphic either to $OJP(n,n)$ or to $OJP(n,n+1)$, for some $n\in\mathbb{Z}_{\geq 0}$. Suppose $OJP\cong OJP(n,n)$ and let us thus identify OJP with $OJP(n,n)$. Then also the corresponding Lie superalgebras $Lie(OJP,\mu')$ and $Lie(OJP(n,n),\mu)$ can be identified, hence there exists an automorphism s of $HO(n+2,n+2)$, preserving the grading of type $(0,\ldots,0,1|0,\ldots,0,-1)$ and the decomposition $OJP(n,n)=PO(n,n)[[x_{n+1}]]+\xi_{n+1}PO(n,n)[[x_{n+1}]]$, such that $s(\mu)=\mu'$. From Example 8.7 one deduces that $\mu'=\mu\varphi$ for some invertible $\varphi\in\mathcal{O}(x_1,\ldots,x_n,\xi_1,\ldots,\xi_n)$ such that $\{\varphi,\varphi\}=0$. It follows that, for $f,g\in OJP$, $f\circ g=[[\mu',f],g]=(-1)^{p(f)+1}\{f,g\}^{\varphi}+2\eta fg$. By Remark 5.2, P is hence gauge equivalent to $PO(n,n)$. A similar argument shows that if OJP is isomorphic to $OJP(n,n+1)$ then P is gauge equivalent to $PO(n,n+1)$.

Finally, c) follows from b). Indeed, if $D^{\varphi}=0$, then $D(\varphi)a=\{\varphi,a\}$ for all a. Letting $a=\varphi$, we get $D(\varphi)=0$, hence $\{\varphi,a\}=0$ for all a, hence $\varphi\in\mathbb{F}$. $\square$

References

[1] A. A. Albert *A structure theory for Jordan algebras,* Ann. of Math. 48 (1947), 545–567.

[2] R. J. Blattner *A theorem of Cartan and Guillemin,* J. Differential Geom. 5 (1970), 295–305.

[3] N. Cantarini, V. G. Kac *Infinite dimensional primitive linearly compact Lie superalgebras,* Adv. Math., 207(1) (2006), 328–419.

[4] N. Cantarini, V. G. Kac *Classification of linearly compact simple Jordan and generalized Poisson superalgebras,* J. Algebra 313 (2007), 100–124.

[5] S.-J. Cheng, V. G. Kac *Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields,* Transf. Groups 4 (1999), 219–272.

[6] A. Dzhumadildaev *Novikov-Jordan algebras,* Comm. in Algebra 30 (2002), 5207–5240.

[7] A. Dzhumadildaev *Anti-commutative algebras with skew-symmetric identities,* J. Algebra Appl. 8 (2009), 157–180.

[8] D. Fattori, V. G. Kac *Classification of finite simple Lie conformal superalgebras,* J. Algebra 258 (2002), 23–59.

[9] V. W. Guillemin *A Jordan-Hölder decomposition for a certain class of infinite-dimensional Lie algebras,* J. Diff. Geom. 2 (1968), 313–345.

[10] V. G. Kac *Lie Superalgebras,* Adv. Math. 26 (1977), 8–96.

[11] V. G. Kac *Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras,* Comm. Algebra 5(13) (1977), 1375–1400.

[12] V. G. Kac *Classification of infinite-dimensional simple linearly compact Lie superalgebras,* Adv. Math. 139 (1998), 1–55.

[13] I. L. Kantor *Jordan and Lie superalgebras determined by a Poisson algebra,* Amer. Math. Soc. Transl. 151 (1992), 55–80.

[14] D. King, K. McCrimmon *The Kantor construction of Jordan superalgebras,* Comm. Algebra 20(1) (1992), 109–126.